Bull. Korean Math. Soc. **35** (1998), No. 1, pp. 53–61

POSITIVE IMPLICATIVE AND ASSOCIATIVE FILTERS OF LATTICE IMPLICATION ALGEBRAS

YOUNG BAE JUN*, YANG XU AND KEYUN QIN

ABSTRACT. We introduce the concepts of a positive implicative filter and an associative filter in a lattice implication algebra. We prove that (i) every positive implicative filter is an implicative filter, and (ii) every associative filter is a filter. We provide equivalent conditions for both a positive implicative filter and an associative filter.

1. Introduction

In order to research the logical system whose propositional value is given in a lattice, Y. Xu [Xu2] proposed the concept of lattice implication algebras, and discussed their some properties in [Xu1] and [Xu2]. Y. Xu and K. Y. Qin [XQ] introduced the notions of a filter and an implicative filter in a lattice implication algebra, and investigated their properties. Y. B. Jun [Ju] gave some equivalent conditions that a filter is an implicative filter in a lattice implication algebra, and established an extension property for implicative filter. In this paper, We introduce the concepts of a positive implicative filter and an associative filter in a lattice implication algebra. We prove that every positive implicative filter is an implicative filter, and hence a filter, and that every associative filter is a filter. We give an example to show that a filter may not be an associative filter. We provide equivalent conditions for both a positive implicative filter and an associative filter.

Received March 10, 1997.

1991 Mathematics Subject Classification: 03G10, 06B10.

Key words and phrases: Lattice implication algebra, positive implicative filter, associative filter.

*Supported by the LG Yonam Foundation, 1995.

2. Preliminaries

DEFINITION 2.1 (Xu [Xu2]). By a *lattice implication algebra* we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution " $'$ " and a binary operation " $\rightarrow$ " satisfying the following axioms for all $x, y, z \in L$:

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(I2) $x \rightarrow x = 1$,
(I3) $x \rightarrow y = y' \rightarrow x'$,
(I4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$,
(I5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
(L1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
(L2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$.

We can define a partial ordering $\leq$ on a lattice implication algebra L by $x \leq y$ if and only if $x \rightarrow y = 1$.

EXAMPLE 2.2. Let $L := \{0, a, b, c, 1\}$. Define the partially ordered relation on L as $0 < a < b < c < 1$, and define $x \wedge y := \min\{x, y\}$, $x \vee y := \max\{x, y\}$ for all $x, y \in L$ and "$'$"and "$\rightarrow$" as follows:

x	x'
0	1
a	c
b	b
c	a
1	0

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	b	c	1	1
1	0	a	b	c	1

Then $(L, \vee, \wedge, ', \rightarrow)$ is a lattice implication algebra.

OBSERVATION (Xu [Xu2]). *In a lattice implication algebra L, the following hold for all $x, y, z \in L$:*

(1) $0 \rightarrow x = 1$, $1 \rightarrow x = x$ *and* $x \rightarrow 1 = 1$,
(2) $x \leq y$ *implies* $y \rightarrow z \leq x \rightarrow z$ *and* $z \rightarrow x \leq z \rightarrow y$,
(3) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$,
(4) $x \rightarrow ((x \rightarrow y) \rightarrow y) = 1$.

In what follows, L would mean a lattice implication algebra unless otherwise specified. In [XQ], Y. Xu and K. Y. Qin defined the notions of a filter and an implicative filter in a lattice implication algebra.

DEFINITION 2.3 (Xu and Qin [XQ]). Let $(L, \vee, \wedge, ', \rightarrow)$ be a lattice implication algebra. A subs F of L is called a *filter* of L if it satisfies for all $x, y \in L$:

(F1) $1 \in F$,

(F2) $x \in F$ and $x \rightarrow y \in F$ imply $y \in F$.

A subset F of L is called an *implicative filter* of L, if it satisfies (F1) and

(F3) $x \rightarrow (y \rightarrow z) \in F$ and $x \rightarrow y \in F$ imply $x \rightarrow z \in F$

for all $x, y, z \in L$.

LEMMA 2.4 (Jun [Ju]). *Let F be a non-empty subset of L. Then F is a filter of L if and only if it satisfies for all $x, y \in F$ and $z \in L$:*

(i) *$x \leq y \rightarrow z$ implies $z \in F$.*

LEMMA 2.5 (Jun [Ju]). *Every filter F of L has the following property:*

$$x \leq y \text{ and } x \in F \text{ imply } y \in F.$$

3. Positive Implicative Filters

MAIN DEFINITION 1. A subset F of L is called a *positive implicative filter* of L if it satisfies (F1) and

(F4) $x \rightarrow ((y \rightarrow z) \rightarrow y) \in F$ and $x \in F$ imply $y \in F$

for all $x, y, z \in L$.

We first give an example of a positive implicative filter of a lattice implication algebra.

EXAMPLE 3.1. Let $L := \{0, a, b, c, d, 1\}$ be a set with Figure 1 as a partial ordering. Define a unary operation " $'$ " and a binary operation " $\rightarrow$ " on L as follows (Tables 1 and 2, respectively):

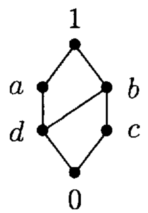

Figure 1

x	x'
0	1
a	c
b	d
c	a
d	b
1	0

Table 1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Table 2

Define $\vee$- and $\wedge$-operations on L as follows:

$$x \vee y := (x \to y) \to y,$$
$$x \wedge y := ((x' \to y') \to y')',$$

for all $x, y \in L$. Then L is a lattice implication algebra. It is easy to check that $F := \{b, c, 1\}$ is a positive implicative filter of L.

THEOREM 3.1. *Every positive implicative filter of L is a filter.*

Proof. Let F be a positive implicative filter of L and let $x \to y \in F$ and $x \in F$ for all $x, y \in L$. Putting $z = y$ in (F4), we have $x \to ((y \to y) \to y) = x \to y \in F$ and $x \in F$. It follows from (F4) that $y \in F$, whence F is a filter of L. □

REMARK 3.2. The converse of Theorem 3.1 may not be true. In fact, consider a lattice implication algebra L as in Example 2.2. We know that $\{1\}$ is a filter of L, but it is not a positive implicative filter, since $1 \to ((c \to a) \to c) = 1 \in \{1\}$ and $c \notin \{1\}$. Also, in Example 3.1, we know that the subset $G := \{a, 1\}$ is a filter, but not a positive implicative filter of L, since $1 \to ((b \to c) \to b) = 1 \in G$ for $b \notin G$.

Now we give an equivalent condition that every filter is a positive implicative filter.

THEOREM 3.3. *Let F be a filter of L. Then F is a positive implicative filter of L if and only if for all $x, y \in L$,*
(F5) $(x \to y) \to x \in F$ *implies* $x \in F$.

Proof. Assume that F is a positive implicative filter of L and let $(x \to y) \to x \in F$ for all $x, y \in L$. Then we have

$$1 \to ((x \to y) \to x) = (x \to y) \to x \in F.$$

Since $1 \in F$, it follows from (F4) that $x \in F$, and (F5) holds.

Conversely, suppose that F satisfies (F5). Let $x \to ((y \to z) \to y) \in F$ and $x \in F$ for all $y, z \in L$. Then $(y \to z) \to y \in F$ by (F2), which implies $y \in F$ by (F5). Hence F is a positive implicative filter of L and the proof is complete. □

THEOREM 3.4. *Let F be a non-empty subset of L. If F is a positive implicative filter of L, then it is an implicative filter of L.*

Proof. Let $x \to (y \to z) \in F$ and $x \to y \in F$ for all $x, y, z \in L$. Then

$$\begin{aligned} x \to (y \to z) &= y \to (x \to z) && \text{[by (I1)]} \\ &\leq (x \to y) \to (x \to (x \to z)). && \text{[by (I1) and (3)]} \end{aligned}$$

Since F is a filter of L (see Theorem 3.1), it follows from Lemma 2.4 that $x \to (x \to z) \in F$. On the other hand, using (I1) and (I5) we have

$$\begin{aligned} &((x \to z) \to z) \to (x \to z) \\ &= x \to (((x \to z) \to z) \to z) \\ &= x \to ((z \to (x \to z)) \to (x \to z)) \\ &= x \to ((x \to (z \to z)) \to (x \to z)) \\ &= x \to ((x \to 1) \to (x \to z)) \\ &= x \to (1 \to (x \to z)) \\ &= x \to (x \to z) \in F. \end{aligned}$$

By Theorem 3.3, we get $x \to z \in F$. This completes the proof. □

OPEN PROBLEM. *Does the converse of* Theorem 3.4 *hold?*

4. Associative Filters

MAIN DEFINITION 2. Let x be a fixed element of L. A subset F of L is called an *associative filter* of L with respect to x if it satisfies (F1) and

(A1) $x \to (y \to z) \in F$ and $x \to y \in F$ imply $z \in F$

for all $y, z \in L$. An associative filter of L with respect to all $x \neq 0$ is called an *associative filter* of L.

Clearly, an associative filter with respect to 0 is the whole algebra L. An associative filter with respect to 1 is coincident with a filter.

EXAMPLE 4.1. Let $L := \{0, a, b, c, d, 1\}$ be a set with Figure 2 as a partial ordering. Define a unary operation " $'$ " and a binary operation " $\to$ " on L by Tables 3 and 4, respectively, and define $\vee$- and $\wedge$-operations on L as follows:

$$x \vee y := (x \to y) \to y \text{ and } x \wedge y := ((x' \to y') \to y')', \text{ respectively,}$$

for all $x, y \in L$.

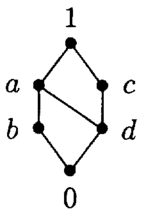

Figure 2

x	x'
0	1
a	d
b	c
c	b
d	a
1	0

Table 3

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

Table 4

Then L is a lattice implication algebra. It is routine to verify that $F := \{1, a, b\}$ is an associative filter of L with respect to a and b, but not with respect to c and d, since

$$c \to (b \to d) = c \to c = 1 \in F,\ c \to b = b \in F \text{ but } d \notin F,$$

and

$$d \to (b \to c) = d \to c = 1 \in F,\ d \to b = a \in F \text{ but } c \notin F.$$

PROPOSITION 4.1. *Every associative filter with respect to x contains x itself.*

Proof. If $x = 0, 1$ then it is trivial. Assume $x \neq 0$. Let F be an associative filter of L with respect to x. Note that $x \to (1 \to x) = x \to x = 1 \in F$ and $x \to 1 = 1 \in F$. It follows from (A1) that $x \in F$. □

THEOREM 4.1. *Every associative filter is a filter.*

Proof. Let F be an associative filter of L and let $x \to y \in F$ and $x \in F$ for all $x, y \in L$. Then $1 \to x = x \in F$ and $1 \to (x \to y) = x \to y \in F$. It follows from (A1) that $y \in F$. Hence F is a filter of L □

REMARK 4.2. The converse of Theorem 4.1 may not be true. In fact, we know that, in Example 2.2, $\{1\}$ is a filter of L. But it is not an associative filter of L, since $a \to (b \to c) = a \to 1 = 1 \in \{1\}$ and $a \to b = 1 \in \{1\}$, but $c \notin \{1\}$.

Now we give equivalent conditions that every filter is an associative filter.

THEOREM 4.3. *Let F be a filter of L. Then F is an associative filter if and only if it satisfies*

(A2) $x \to (y \to z) \in F$ *implies* $(x \to y) \to z \in F$

for all $x, y, z \in L$.

Proof. If a filter F of L satisfies the condition (A2), then clearly F is associative. Conversely, let F be an associative filter of L and let $x \to (y \to z) \in F$ for all $x, y, z \in L$. Then

$$\begin{aligned} & x \to ((y \to z) \to ((x \to y) \to z)) \\ &= (y \to z) \to (x \to ((x \to y) \to z)) && \text{[by (I1)]} \\ &= (y \to z) \to ((x \to y) \to (x \to z)) && \text{[by (I1)]} \\ &= 1 \in F, && \text{[by (I1) and (3)]} \end{aligned}$$

which implies from (A1) that $(x \to y) \to z \in F$. This completes the proof. □

THEOREM 4.4. *Let F be a filter of L. Then F is an associative filter if and only if it satisfies*

(A3) $x \to (x \to y) \in F$ *implies* $y \in F$

for all $x, y \in L$.

Proof. Let F be a filter of L. It is sufficient to show that (A2) and (A3) are equivalent. Putting $x = y$ in (A2) and using (I2) and (1), we obtain (A3). Assume that (A3) holds and let $x \to (y \to z) \in F$ for all $x, y, z \in L$. Using (I1), (I2), (1), (2) and (3), we have

$$\begin{aligned}
&(x \to (y \to z)) \to (x \to (x \to ((x \to y) \to z)))\\
&= 1 \to ((x \to (y \to z)) \to (x \to (x \to ((x \to y) \to z))))\\
&= ((y \to z) \to (x \to ((x \to y) \to z))) \to ((x \to (y \to z))\\
&\qquad \to (x \to (x \to ((x \to y) \to z))))\\
&= (x \to (y \to z)) \to (((y \to z) \to (x \to ((x \to y) \to z)))\\
&\qquad \to (x \to (x \to ((x \to y) \to z))))\\
&\geq (x \to (y \to z)) \to (x \to (y \to z)) = 1,
\end{aligned}$$

which implies that

$$(x \to (y \to z)) \to (x \to (x \to ((x \to y) \to z))) = 1 \in F.$$

Since $x \to (y \to z) \in F$ and since F is a filter, it follows that

$$x \to (x \to ((x \to y) \to z)) \in F.$$

By using (A3), we conclude that $(x \to y) \to z \in F$. This completes the proof. □

ACKNOWLEDGEMENTS. The authors express their thanks to the referee for his/her valuable suggestions.

References

[Ju] Y. B. Jun, *Implicative filters of lattice implication algebras*, Bull. Korean Math. Soc. **34** (1997), 193-198.

[JX] Y. B. Jun and Y. Xu, *Fuzzy filters of lattice implication algebras*, submitted.
[Xu1] Y. Xu, *Homomorphisms in lattice implication algebras*, Proc. of 5th Many-Valued Logical Congress of China (1992), 206-211.
[Xu2] ———, *Lattice implication algebras*, J. of Southwest Jiaotong Univ. **1** (1993), 20-27.
[XQ] Y. Xu and K. Y. Qin, *On filters of lattice implication algebras*, J. Fuzzy Math. **1** (1993), 251-260.
[YW] B. Yuan and W. Wu, *Fuzzy ideals on a distributive lattice*, Fuzzy Sets and Systems **35** (1990), 231-240.

YOUNG BAE JUN
DEPARTMENT OF MATHEMATICS, GYEONGSANG NATIONAL UNIVERSITY, CHINJU 660-701, KOREA
E-mail: ybjun@nongae.gsnu.ac.kr

YANG XU
DEPARTMENT OF APPLIED MATHEMATICS, SOUTHWEST JIAOTONG UNIVERSITY, CHENGDU, SICHUAN 610031, P. R. CHINA
E-mail: yxu@center2.swjtu.edu.cn

KEYUN QIN
DEPARTMENT OF MATHEMATICS, HENAN NORMAL UNIVERSITY, XINXIANG, HENAN 453002, P. R. CHINA