Effect of Boundary Condition History on the Symmetry Breaking Bifurcation of Wall-Driven Cavity Flows

Ji Ryong Cho*
Department of Mechanical and Automotive Engineering, Inje University, 607 Obangdong, Kimhae 621-749, Korea

A symmetry breaking nonlinear fluid flow in a two-dimensional wall-driven square cavity taking symmetric boundary condition after some transients has been investigated numerically. It has been shown that the symmetry breaking critical Reynolds number is dependent on the time history of the boundary condition. The cavity has at least three stable steady state solutions for Re=300−375, and two stable solutions if Re>400. Also, it has also been showed that a particular solution among several possible solutions can be obtained by a controlled boundary condition.

Key Words : Cavity, Boundary Condition History, Symmetry Breaking Bifurcation, Multiplicity, Critical Reynolds number

1. Introduction

The nonlinear hydrodynamic convection produces bifurcating fluid flows if the flow inertia is sufficiently larger than the viscous friction. Among various bifurcating phenomena, a Pitchfork bifurcation, or, a formation of steady, symmetry-broken fluid flow field under a symmetric boundary condition, is the subject of this study. In most high Reynolds number engineering flow, this sort of bifurcation draws little attention because the flow is usually turbulent. In a turbulent flow, a local phenomenon is not isolated within a limited region because of large scale energetic turbulent eddies. Instead, it spreads over a large space, and the asymmetry smears out. However, inside a MEMS fluidic device, a working fluid sometimes has small characteristic length and velocity scales to yield Re=$\vartheta(10)\sim\vartheta(100)$. In such a Reynolds number range, the possibility of Pitchfork bifurcation should be investigated with a special attention because it greatly changes mixing and heat transfer characteristics inside the fluidic device. Symmetry breaking bifurcations have been observed experimentally and numerically in several flow geometries : in the sudden expansion channel by Durst et al. (1974) ; in the twin jet confined with side walls by Soong et al. (1998) ; in the counter-flow jet by Salinger et al. (2001) ; and, around a circular cylinder confined in the channel by Sahin and Owens (2004). All these researches report the critical Reynolds number for the transition from symmetric to asymmetric state. However, neither of them considered the effect of boundary and/or initial conditions at all. Presumably, the comment of Drikakis (1997) is the only one about this issue. He noted that the computed critical Reynolds number for the symmetry breaking bifurcation of the sudden expansion channel is different from that of Durst et al. (1974), and supposed that the difference of the initial condition between the two researches is responsible for the discrepancy. In this study, we are going to address the effect of boundary condition change history on the Pitchfork bifurcation inside a square cavity.

* **E-mail** : mechjrc@ijnc.inje.ac.kr
TEL : +82-55-320-3187; **FAX** : +82-55-324-1723
Department of Mechanical and Automotive Engineering, Inje University, 607 Obangdong, Kimhae 621-749, Korea. (Manuscript **Received** April 13, 2005; **Revised** August 29, 2005)

2. Bifurcation in a Cavity

Although the experimental realization is impossible, a cavity flow is an excellent flow model in understanding complex fluid flow phenomena. A cavity has simple geometry and well-defined boundary conditions, but the fluid flow shows separation and reattachment of fluid stream and formation of vortical structures. Because of the simple geometry and the complex flow phenomena, the cavity flow serves as a benchmark flow. Ghia et al. (1982) adopted the cavity flow in order to validate their numerical scheme. Whereas, Choi et al. (2004) employed the cavity flow in order to evaluate the relative performance of several turbulence models. Bifurcation of a square cavity driven by a single, top-wall has been numerically studied by Goodrich et al. (1990) and Shen (1991). They observed a persisting oscillation of velocity components under stationary boundary condition when the flow Reynolds number is high enough. Comprehensive and in-depth numerical study of rectangular cavity driven by two opposing side-walls has been done by Albensoeder et al. (2001). Their computations has been done on a three dimensional parameter-space (Re_1, Re_2, Γ), where both Re_1 and Re_2 are Reynolds numbers based on the two moving wall and Γ is the aspect ratio of the cavity. The continuation method has been used to track the bifurcating solution on the three dimensional parameter-space, or the solution at a specific point $\lambda(Re_1, Re_2, \Gamma)$ has been used as an initial guess at a new point $\lambda+\delta\lambda$. They identified multiple steady state solutions under a fixed boundary condition. However, in a practical (engineering) sense, multiplicity of solution causes confusion because they don't know which solution will be yielded among several possibilities. It is clear that initial and/or boundary conditions contribute to the nonlinear process and finally to the steady state solution. As far as the present author's knowledge, no such works have been reported in technical periodicals. Present research address this issue for a cavity shown in Fig. 1. The cavity is driven by two neighboring

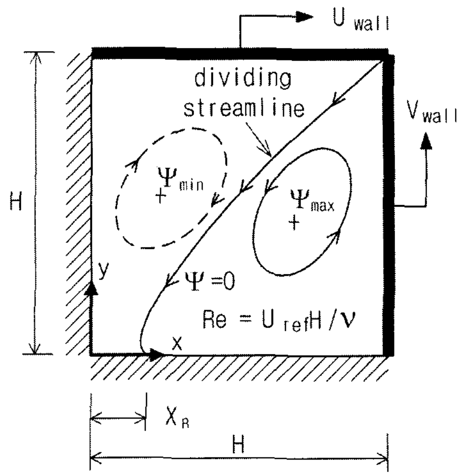

Fig. 1 Configuration of the cavity

wall which are moving towards their joint corner. Clearly, a diagonally symmetric pattern is expected at a sufficiently low Reynolds number. But the symmetry is broken if the nonlinear convective process dominates the viscous diffusive process (for example, at Re=400, Cho, 1999). In this study, in order to investigate the effect of temporal development history of boundary condition on the steady state solution, two different boundary conditions are applied :

The symmetric boundary condition case,

$$U_{wall}=V_{wall}=U_{ref},\ t>0 \tag{1}$$

The asymmetric boundary condition case,

$$U_{wall}=U_{ref},\ t>0$$
$$V_{wall}=\begin{cases} 0, & t<\tau \\ U_{ref} & t\geq\tau \end{cases} \tag{2}$$

Or, the two moving walls start to accelerate simultaneously for the symmetric boundary condition case. Whereas the top wall starts to accelerate first and the right side wall accelerates after time interval τ for the asymmetric boundary condition case. But the two cases have the same condition for $t\geq\tau$. It is assumed that the flow is at rest initially for both cases.

3. Numerical Methods

The flow inside the cavity is unsteady, 2-dimensional, and incompressible. We adopt the traditional vorticity-stream function formulation is preferred to velocity-based ones because the pressure term disappears naturally. In this study, however, instead of solving two separate equations for vorticity and stream function, the two equations are coupled to yield the so-called pure stream function formulation (Canuto et al., 1988):

$$\frac{\partial \Delta \Psi}{\partial t} = G(\Psi, Re) \tag{3}$$

where

$$G = \frac{\Delta^2 \Psi}{Re} - \frac{\partial \Psi}{\partial y}\frac{\partial \Delta \Psi}{\partial x} + \frac{\partial \Psi}{\partial x}\frac{\partial \Delta \Psi}{\partial y} \tag{4}$$

The symbols Ψ and Δ denote the stream function and the Laplacian operator, respectively. All variables are normalized with U_{REF} and H, and Re is the Reynolds number. To drive the numerical solution from the stagnant initial state to the developed steady state the following second order accurate scheme is employed. The Crank-Nicolson scheme is adopted for time integration:

$$\Delta \Psi^{n+1,k+1} = 0.5\delta t(G^{n+1,k} + G^n) + \Delta \Psi^n \tag{5}$$

where superscripts n and k are time level and sub-iteration index, respectively. A system matrix is constructed with the Laplacian operator, and the iterative solution is easily obtained by applying the simple tri-diagonal matrix algorithm at a fixed time level until the solution is converged. The second order accurate central differencing is adopted for spatial gradient terms. Therefore the interior solution is expected to be second order accurate both in time and space. Along flow boundaries, $\Delta\Psi$ as well as Ψ itself should be specified. Relevant second order accurate boundary conditions along the north wall, for example, are

$$\Psi_{i,NJ}^{n+1,k+1} = 0 \tag{6}$$

$$\Delta \Psi_{i,NJ}^{n+1,k+1} = \frac{8\Psi_{i,NJ-1}^{n+1,k} - \Psi_{i,NJ-2}^{n+1,k} + 6hu_{i,NJ}^{n+1}}{2h^2} \tag{7}$$

where h is the grid spacing. Four grids near the wall are distributed with uniform spacing, and it is gradually increased towards the center of the cavity. For all computations in this study, the grid number is 81×81, and the Courant number, $C = U_{REF}\delta t/h$, varies between $0.164-1.000$ within the cavity. To verify present numerical method, the usual top-lid driven cavity of Ghia et al. (1982) has been computed. The sub-iteration has been continued until $|\Psi^{n+1,k+1} - \Psi^{n+1,k}|_{\max} < 10^{-7}$. Ghia et al. adopted very fine grid and reported the strength of the primary recirculating cell in terms of streamfunction: it was -0.1139 and -0.1179 at Re=400 and 1000, respectively. Present computations returned -0.11325 and -0.11727 for each case. Therefore, the numerical error is 0.57% at Re=400 and 0.54% at Re=1000. Admitting this level of numerical error, we have computed the present cavity for Re=100−1000. In asymmetric boundary condition cases, the time interval between accelerations, τ, is set at 100 for all Reynolds numbers studied here. Preliminary computations with only one moving wall yield a quasi steady state after t=100 for all cases. In fact, the computation with the asymmetric boundary condition can be considered as a computation with the symmetric boundary condition but with a nonzero, different initial condition. To obtain steady state solutions, time marching is continued until there are no changes in $\Psi_{\max}$ and $\Psi_{\min}$.

4. Results

Typical steady state fluid flow fields are shown in Fig. 2. At Re=100, both the symmetric boundary condition case, Re100SM, and the asymmetric boundary condition case, Re100AS, yield identical, diagonally symmetric flow pattern. Besides the two big counter rotating bubbles, two tiny counter rotating bubbles are formed near the bottom left corner of the cavity. Thereby, the fluid stream separated at top right corner does not arrive at bottom left corner but away from it. The size of the two corner bubbles at Re=200 is decreased with the increase of the Reynolds number for both Re200SM and Re200AS: At higher Reynolds number, the increased momentum of fluid moving from top right corner towards bottom left corner compresses the corner bubbles more tightly. Further increase of Reynolds number yields a drastic change of flow pattern above a certain Reynolds number. For the AS case, the change occurs between Re=275 and 300 (Figs. 2 and 3). Similar change also occurs for the SM case, but the change occurs at much higher Reynolds number between Re=375 and 400. At Re=300, the diagonal symmetry is maintained for Re300SM, but broken for Re300AS. This result clearly shows that the symmetry breaking critical Reynolds number is strongly dependent on the time history of the boundary condition. Though it is not reported here, if we change the acceleration sequence of two moving walls of Re300AS, a diagonally mirrored flow pattern is obtained. Thus, there exist, at least, three different stable steady state solutions between Re=300−375: one symmetric solution and two asymmetric, mirror-imaged solutions. Above Re=400, both SM and AS cases have indistinguishable, virtually the same, asymmetric solution. It seems that the symmetric flow pattern is unstable. And, only the two mirror-imaged asymmetric solutions could be obtained. Vorticity values at the center of two counter rotating big bubbles are plotted in Fig. 4. For both SM and AS cases, the bifurcation is produced, and consistent with the result of Figs. 2 and 3. Interesting point is that the bifurca-

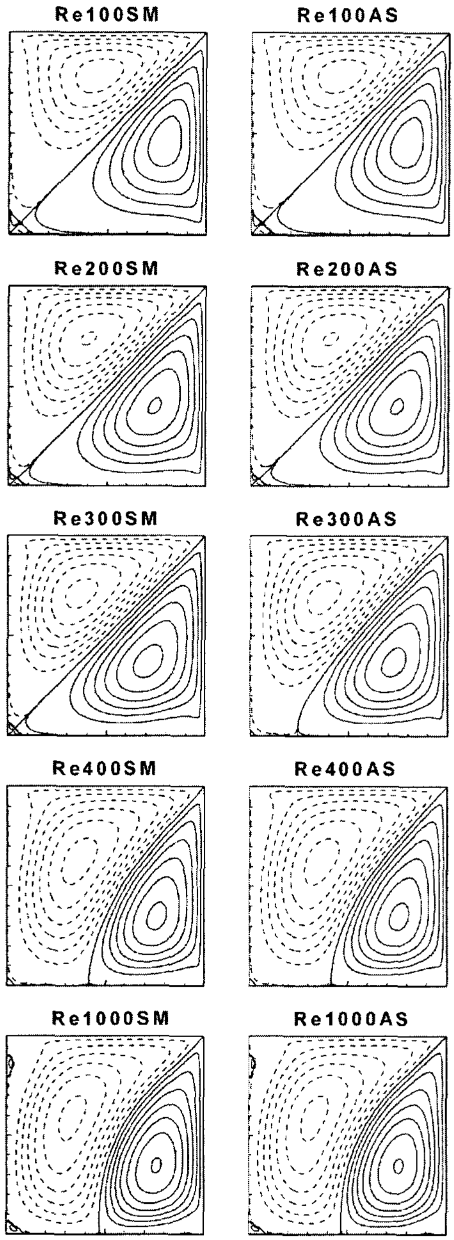

Fig. 2 Steady state streamline patterns

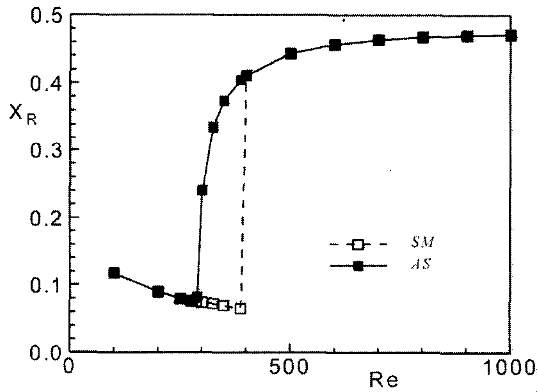

Fig. 3 Bifurcation of the reattachment length

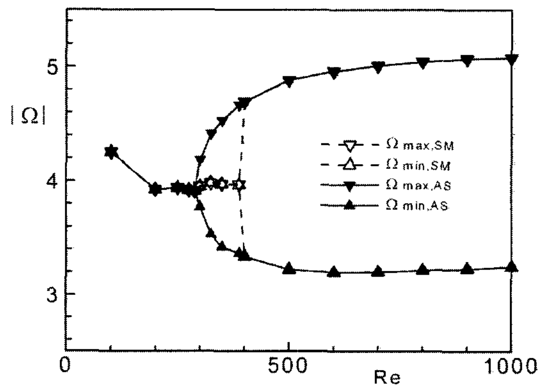

Fig. 4 Bifurcation of the vorticity at the center of recirculating bubbles

tion path of vorticity : The symmetry breaking bifurcation occurs suddenly above the critical Reynolds number for SM case, but the change is gradual for AS case. Notable thing is the slight, wavy variation before the bifurcation. However, unfortunately, the present author could not explain this behavior. It seems that this is not due to a numerical imperfection of present method : Repeated computations with much more tight criterions for the convergence of sub-iteration and steady state returned the same behavior.

5. Summary

In this study, it is observed numerically that the symmetry breaking critical Reynolds number is strongly dependent on the time history of the boundary condition, and that there exist at least three stable steady state solutions between Re= 300−375, and two stable steady state solutions for Re>400. Multiplicity of fluid flow may cause unexpected behavior for a small sized machine in which the flow Reynolds number is not high enough. In this study, it is also showed that a particular solution among several possible solutions can be obtained by a controlled boundary condition. Devising a control or trigger method to jump from a particular solution to other one is supposed to be a challenging area.

Acknowledgments

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