THE AP-HENSTOCK EXTENSION OF THE DUNFORD AND PETTIS INTEGRALS

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Abstract. In this paper, we introduce the AP-Henstock Dunford, AP-Henstock Pettis and AP-Henstock Bochner integral Banach-valued functions and investigate some properties of the these integrals.

1. Introduction and preliminaries

The Denjoy-Dunford, Denjoy-Pettis, and Denjoy-Bochner integral are the extension of Dunford, Pettis, and Bocher integrals, respectively. These integrals were defined and studied by Gordon [4]. He showed that Denjoy-Dunford(Denjoy-Bochner) integrable function on \([a, b]\) is Dunford(Bochner) integrable on some subinterval of \([a, b]\) and that for spaces that do not contain copy \(c_0\), a Denjoy-Pettis integrable function on \([a, b]\) is Pettis integrable on some subinterval of \([a, b]\). In 2000, Park [5] introduced the Denjoy extension of the Riemann and McShane integral and proved some properties of these integral.

In this paper, we define and study the AP-Henstock extension of Dunford, Pettis, and Bochner integrals of functions mapping \([a, b]\) into Banach space \(X\), respectively.

Throughout this paper, \(X\) will denote a real Banach space and \(X^*\) its dual.

Let \(E\) be measurable set and let \(c\) be a real number. The density of \(E\) at \(c\) is defined by...
\[ d_s E = \lim_{h \to 0^+} \frac{\mu(E \cap (c - h, c + h))}{2h}, \]

provided the limit exists. The point \( c \) is called a point of density of \( E \) if \( d_s E = 1 \). The \( E^d \) represents the set of all point \( x \in E \) such that \( x \) is a point of density of \( E \).

A function \( F : [a, b] \to R \) is said to be approximately differentiable at \( c \in [a, b] \) if there exists a measurable set \( E \subset [a, b] \) such that \( c \in E^d \) and
\[ \lim_{x \to c} \frac{f(x) - F(c)}{x - c} \]
exists. The approximate derivative of \( F \) at \( c \) is denoted by \( F'_ap(c) \).

An approximate neighborhood (or ap-nbd) of \( x \in [a, b] \) is a measurable set \( S_x \subset [a, b] \) containing \( x \) as a point of density. For every \( x \in E \subset [a, b] \), choose an ap-nbd \( S_x \subset [a, b] \) of \( x \), then we say that \( S = \{ S_x : x \in E \} \) is a choice on \( E \). A tagged interval \( (x, [c, d]) \) is said to be subordinate to the choice \( S = \{ S_x \} \) if \( c, d \in S_x \). Let \( P = \{ (x_i, [c_i, d_i]) : 1 \leq i \leq n \} \) be a finite collection of non-overlapping tagged intervals. If \( (x_i, [c_i, d_i]) \) is subordinate to a choice \( S \) for each \( i \), then we say that \( P \) is subordinate to \( S \). If \( P \) is subordinate to \( S \) and \( [a, b] = \bigcup_{i=1}^n [c_i, c_{i+1}] \), then we say that \( P \) is a tagged partition of \( [a, b] \) that is subordinate to \( S \).

**Definition 1.1.** A function \( f : [a, b] \to X \) is AP-Henstock integrable on \([a, b]\) if there exists a vector \( A \in X \) with the following property: for each \( \epsilon > 0 \) there exists a vector \( A \in X \) such that \( \| f(P) - A \| < \epsilon \) whenever \( P \) is a tagged partition of \([a, b]\) that is subordinate to \( E \) where \( f(P) = (P) \sum f(x) \mid I \mid \). The vector \( A \) is called the AP-Henstock integral of \( f \) on \([a, b]\) and denoted by \( \int_a^b f \).

Recall that \( F : [a, b] \to R \) is \( AC_G \) on a measurable set \( E \subset [a, b] \) if for each \( \epsilon > 0 \) there exists a position number \( \eta \) and a choice \( S \) on \( E \) such that \( \|(P) \sum F(I)\| < \epsilon \) for every finite collection \( P \) of non-overlapping tagged intervals that is subordinate to \( S \) and satisfies \( (P) \sum \mid I \mid < \eta \), where \( \mid I \mid \) is the Lebesgue measure of an interval \( I \). The function \( F \) is \( AC_G \) on \( E \) if \( E \) can be expressed as a countable union of measurable sets on each of which \( F \) is \( AC_G \).

**Definition 1.2.** ([6]) A function \( f : [a, b] \to X \) is AP-Denjoy integrable on \([a, b]\) if there exists an \( AC_G \) function \( F \) on \([a, b]\) such that \( F'_ap = f \) almost everywhere on \([a, b]\).

**Theorem 1.3.** ([6]) A function \( f : [a, b] \to X \) is AP-Denjoy integrable on \([a, b]\) if and only if \( f \) is AP-Henstock integrable on \([a, b]\).
DEFINITION 1.4. ([3])

(a) A function \( f : [a, b] \to X \) is Denjoy-Dunford integrable on \([a, b]\) if for each \( x^* \) in \( X^* \) the function \( x^* f \) is Denjoy integrable on \([a, b]\) and if for every interval \( I \) in \([a, b]\) there exists a vector \( x^{**} \) in \( X^{**} \) such that \( x^{**}(x^*) = \int_I x^* f \) for \( x^* \) in \( X^* \).

(b) A function \( f : [a, b] \to X \) is Denjoy-Pettis integrable on \([a, b]\) if \( f \) is Denjoy-Dunford integrable on \([a, b]\) and if \( x^{**} \) \( I \) \( x^{**} \) in \( X^{**} \) for every interval \( I \) is \([a, b]\).

(c) A function \( f : [a, b] \to X \) is Denjoy-Bochner integrable on \([a, b]\) if there exists an \( ACG \) function \( F : [a, b] \to X \) such that \( F \) is approximately differentiable almost everywhere on \([a, b]\) and \( F'_{ap} = f \) almost everywhere on \([a, b]\).

2. AP-Henstock-Dunford and AP-Henstock-Pettis integrability

we introduce the AP-Henstock-Dunford and AP-Henstock-Pettis integral of which is extension for Denjoy-Dunford and Denjoy-Pettis integral and investigate some properties of there integrals.

DEFINITION 2.1. (a) A function \( f : [a, b] \to X \) is AP-Henstock-Dunford integrable on \([a, b]\) if for each \( x^* \) in \( X^* \) the function \( x^* f \) is AP-Henstock integrable on \([a, b]\) and if for every interval \( I \) in \([a, b]\) there exists a vector \( x^{**} \) in \( X^{**} \) such that \( x^{**}(x^*) = \int_I x^* f \) for all \( x^* \) in \( X^* \).

(b) A function \( f : [a, b] \to X \) is AP-Henstock-Pettis integrable on \([a, b]\) if \( f \) is AP-Henstock-Dunford integrable on \([a, b]\) and \( x^{**} \) \( I \) \( x^{**} \) in \( X^{**} \) for every interval \( I \) in \([a, b]\).

(c) A function \( f : [a, b] \to X \) is AP-Henstock-Bochner integrable on \([a, b]\) if there exists an \( ACG \) function \( F : [a, b] \to X \) such that \( F \) is approximately differentiable almost everywhere on \([a, b]\) and such that \( F'_{ap} = f \) almost everywhere on \([a, b]\).

Throughout this paper, \( \langle APD \rangle \int_a^b f \) and \( \langle APP \rangle \int_a^b f \) will denote the AP-Henstock-Dunford integral and the AP-Henstock-Pettis integral of \( F \) on \([a, b]\).

The following theorem was proved by J. I. Games and J. Mendoza [2].
Theorem 2.2. A function \( f : [a, b] \rightarrow X \) is Denjoy-Dunford integrable on \([a, b]\) if and only if \( x^* f \) is Denjoy integrable on \([a, b]\) for each \( x^* \) in \( X^* \).

The following theorems can easily obtained by definition 2.1 and theorem 1.3.

Theorem 2.3. (a) A function \( f : [a, b] \rightarrow X \) is Denjoy-Dunford integrable on \([a, b]\), then \( f \) is AP-Henstock-Dunford integrable on \([a, b]\).

(b) A function \( f : [a, b] \rightarrow X \) is Denjoy-Pettis integrable on \([a, b]\), then \( f \) is AP-Henstock-Pettis integrable on \([a, b]\).

Theorem 2.4. (a) A function \( f : [a, b] \rightarrow X \) is AP-Henstock-Dunford integrable on \([a, b]\), then \( f \) is weakly measurable on \([a, b]\).

(b) A function \( f : [a, b] \rightarrow X \) is a bounded and AP-Henstock-Dunford integrable on \([a, b]\), then \( f \) is Dunford integrable on \([a, b]\).

Proof. Let \( f : [a, b] \rightarrow X \) be AP-Henstock-Dunford integrable on \([a, b]\). Then \( x^* f \) is AP-Henstock integrable on \([a, b]\) for all \( x^* \) in \( X^* \). Hence \( x^* f \) is measurable\([4]\), theorem 16.14 (d)]. Hence \( f \) is weakly measurable. (b) Let \( f : [a, b] \rightarrow X \) be a bounded and AP-Henstock-Dunford integrable on \([a, b]\) for \( x^* f \) is Lebesgue integrable\([4]\), theorem 16.15 (a)]. Therefore \( f \) is Dunford integrable on \([a, b]\).

The next corollary follows immediately from Pettis Measurability and Theorem 2.3.

Corollary 2.5. If \( X \) is a separable Banach space and if \( f : [a, b] \rightarrow X \) is AP-Henstock-Dunford integrable on \([a, b]\). Then \( f \) is measurable.

Theorem 2.6. A function \( f : [a, b] \rightarrow X \) is AP-Henstock-Dunford integrable on \([a, b]\) if and only if \( x^* f \) is AP-Henstock integrable on \([a, b]\) for each \( x^* \) in \( X^* \).

Proof. If a function \( f : [a, b] \rightarrow X \) is AP-Henstock-Dunford integrable on \([a, b]\). By definition, \( x^* f \) is AP-Henstock integrable on \([a, b]\) for each \( x^* \) in \( X^* \). Conversely, if \( x^* f \) is AP-Henstock integrable on \([a, b]\) for each \( X^* \). Let \( B = \{x^* : || x^* || \leq 1 \} \) and for each positive integer \( n \) and let \( V_n = \{x^* \in B : \int_b^a | x^* f | \leq n \} \). Then \( B = \bigcup_n V_n \) and we show next that each \( V_n \) is closed. Let \( y^* \) be a limit point of \( V_n \) and let \( \{x^*_k\} \) be a sequence in \( V_n \) that converges to \( y^* \). Since \( x_k \ast f , x^* f \) are AP-Henstock integrable, \( x_k \ast f , x^* f \) are measurable functions\([4]\), theorem 16.14 (d)].
Also, the sequence \(|x^*_k f|\) converges pointwise to \(|y^* f|\). By the Fatou's Lemma, we have
\[
\int_a^b |x^* f| \leq \liminf_{k \to \infty} \int_a^b |x_k^* f| \leq n
\]

Hence, \(y^* \in V_n\) and we conclude that \(V_n\) is closed. By the Baire Category theorem, there exist an integer \(N\), a real number \(r > 0\), and a vector \(x_0^* \in B\) such that \(\{x^* : \|x^* - x_0^*\| \leq r\} \subset V_N\). For \(x^* \in B\),
\[
\sup_B |\int_E x^* f| \leq \sup_B \int_E |x^* f| \leq \sup_B \int_a^b |x^* f| \leq \frac{2N}{r},
\]
the linear functional \(T_B\) is bounded and hence \(T_E \in X^{**}\).

Theorem 2.7. A function \(f : [a, b] \to X\) os AP-Henstock-Dunford integrable on each interval \([c, d] \subset (a, b)\). If \(\lim_{\substack{c \to a+ \\ d \to b-}} (APD) \int_c^d f\) exists in \(X^{**}\), then \(f\) is AP-Henstock-Dunford integrable on \([a, b]\) and \((APD) \int_a^b = \lim_{\substack{c \to a+ \\ d \to b-}} (APD) \int_c^d f\)

Proof. Let \(x_0^{**} = \lim_{\substack{c \to a+ \\ d \to b-}} (APD) \int_c^d f\). By hypothesis, for each \(x^* \in X^*\), \(x^* f : [a, b] \to R\) is AP-Henstock integrable on each interval \([c, d] \subset (a, b)\) and
\[
<x^*, x_0^{**}> = \lim_{\substack{c \to a+ \\ d \to b-}} <x^*, (APD) \int_c^d f> = \lim_{\substack{c \to a+ \\ d \to b-}} \int_c^d x^* f.
\]

Hence for each \(x^* \in X^*\), \(x^* f\) is AP-Henstock integrable on \([a, b]\). Thus \(f\) is AP-Henstock-Dundford integrable on \([a, b]\) by theorem 2.6 and
\[
<x^*, x_0^{**}> = \lim_{\substack{c \to a+ \\ d \to b-}} <x^*, (APD) \int_c^d f> = <x^*, (APD) \int_a^b f>
\]
for all \(x^* \in X^*\). Hence \((APD) \int_a^b f = x_0^{**} = \lim_{\substack{c \to a+ \\ d \to b-}} (APD) \int_c^d f\)

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