APPLICATION OF $\tilde{g}_\alpha$-CLOSED SETS

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Abstract. The notion of $\tilde{g}_\alpha$ – closed sets in a topological space introduced by R. Devi and A. Selvakumar [2]. In this paper, we introduce the concept of $\tilde{g}_\alpha$-US spaces by utilizing $\tilde{g}_\alpha$ – open sets and study the basic properties of this space.

1. Introduction

In 1967, A. Wilansky [7] introduced and studied the concept of US-spaces. Also, the notion of $\tilde{g}_\alpha$-closed sets of a topological space are discussed by R. Devi et. al. [2]. The concept of slightly continuous functions are introduced and investigated by R. C. Jain [4].

The aim of this paper is to introduce the notion of slightly $\tilde{g}_\alpha$-continuous functions and $\tilde{g}_\alpha$-US spaces. Further, the basic properties of slightly $\tilde{g}_\alpha$-continuous functions are derived. Also we studied the concepts of $\tilde{g}_\alpha$-spaces, $\tilde{g}_\alpha$-convergence, sequentially $\tilde{g}_\alpha$-compactness, sequentially $\tilde{g}_\alpha$-continuity and sequentially $\tilde{g}_\alpha$-sub-continuity.

Throughout the present paper, $X$ and $Y$ are always topological spaces. Let $A$ be a subset of $X$. We denote the interior and the closure of a set $A$ by $\text{int}(A)$ and $\text{cl}(A)$ respectively.

A subset $A$ of a space $X$ is said to be $\alpha$-open [5] if $A \subseteq \text{int} (\text{cl} (\text{int} (A)))$. A subset $A$ of a space $X$ is said to be $\tilde{g}_\alpha$-closed [2] if $\text{acl} (A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open. The complement of a $\tilde{g}_\alpha$-closed set is said to be $\tilde{g}_\alpha$-open. The intersection of all $\tilde{g}_\alpha$-closed sets of $X$ containing $A$ is called $\tilde{g}_\alpha$-closure of $A$ and is denoted by $\tilde{g}\text{acl}(A)$. The union of all $\tilde{g}_\alpha$-open sets of $X$ contained in $A$ is called $\tilde{g}_\alpha$-interior of $A$ and is denoted by $\tilde{g}\text{int}(A)$.

Received June 18, 2010; Revised November 09, 2010; Accepted December 13, 2010.

2010 Mathematics Subject Classification: Primary 54A08.
Key words and phrases: $\tilde{g}_\alpha$ – open set, $\tilde{g}_\alpha$ – US space.
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The family of all $\alpha$-open (resp. $\tilde{g}\alpha$-open, $\tilde{g}\alpha$-closed, clopen, $\tilde{g}\alpha$-clopen) set of $X$ is denoted by $\alpha\textit{O}(X)$ (resp. $\tilde{g}\alpha\textit{O}(X)$, $\tilde{g}\alpha\textit{C}(X)$, $\textit{CO}(X)$, $\tilde{g}\alpha\textit{CO}(X)$).

**Definition 1.1.** ([2]) A function $f : X \to Y$ is $\tilde{g}\alpha$-continuous if $f^{-1}(V)$ is $\tilde{g}\alpha$-open set in $X$ for each open set $V$ of $Y$.

**Definition 1.2.** ([4]) A function $f : X \to Y$ is slightly-continuous if $f^{-1}(V)$ is open set in $X$ for each clopen set $V$ of $Y$.

2. $\tilde{g}\alpha$-US spaces

**Definition 2.1.** A sequence $\{x_n\}$ in a space $X$, $\tilde{g}\alpha$-converges to a point $x \in X$ if $\{x_n\}$ is eventually in every $\tilde{g}\alpha$-open set containing $x$.

**Definition 2.2.** A space $X$ is said to be $\tilde{g}\alpha$-US if every sequence in $X$, $\tilde{g}\alpha$-converges to a point of $X$.

**Definition 2.3.** A space $X$ is said to be

(i) $\tilde{g}\alpha$-$T_1$ if each pair of distinct points $x$ and $y$ in $X$ there exists a $\tilde{g}\alpha$-open set $U$ in $X$ such that $x \in U$ and $y \notin U$ and a $\tilde{g}\alpha$-open set $V$ in $X$ such that $y \in V$ and $x \notin V$.

(ii) $\tilde{g}\alpha$-$T_2$ if for each pair of distinct points $x$ and $y$ in $X$ there exist $\tilde{g}\alpha$-open sets $U$ and $V$ such that $U \cap V = \phi$ and $x \in U, y \in V$.

**Theorem 2.4.** Every $\tilde{g}\alpha$-US-space is $\tilde{g}\alpha$-$T_1$.

*Proof.* Let $X$ be a $\tilde{g}\alpha$-US-space and $x, y$ be two distinct points of $X$. Consider the sequence $\{x_n\}$, where $x_n = x$ for any $n \in N$. Clearly $\{x_n\}$ $\tilde{g}\alpha$-converges to $x$. Since $x \neq y$ and $X$ is $\tilde{g}\alpha$-US, $\{x_n\}$ does not $\tilde{g}\alpha$-converges to $y$, i.e., there exists a $\tilde{g}\alpha$-open set $U$ containing $x$ but not $y$. Similarly, we obtain a $\tilde{g}\alpha$-open set $V$ containing $y$ but not $x$. Thus, $X$ is $\tilde{g}\alpha$-$T_1$.

**Theorem 2.5.** Every $\tilde{g}\alpha$-$T_2$ space is $\tilde{g}\alpha$-US.

*Proof.* Let $X$ be a $\tilde{g}\alpha$-$T_2$ space and $\{x_n\}$ a sequence in $X$. Assume that $\{x_n\}$ $\tilde{g}\alpha$-converges to two distinct points $x$ and $y$. Then $\{x_n\}$ is eventually in every $\tilde{g}\alpha$-$T_2$ then $\{x_n\}$ is eventually in two disjoint $\tilde{g}\alpha$-open sets. This is a contradiction. Therefore, $X$ is $\tilde{g}\alpha$-US.

**Definition 2.6.** A subset $A$ of a space $X$ is said to be

(i) sequentially $\tilde{g}\alpha$-closed if every sequence in $A$ $\tilde{g}\alpha$-converges to a point in $A$. 


(ii) sequentially $\tilde{g}αO$-compact if every sequence in $A$ has a subsequence which $\tilde{g}α$-converges to a point in $A$.

**Theorem 2.7.** A space is $\tilde{g}α-US$ if and only if the diagonal set $\Delta$ is a sequentially $\tilde{g}α$-closed subset of the product space $X \times X$.

**Proof.** Suppose that $X$ is a $\tilde{g}α-US$ space and $\{(x_n, x_n)\}$ is a sequence in the diagonal $\Delta$. It follows that $\{x_n\}$ is a sequence in $X$. Since $X$ is $\tilde{g}α-US$, the sequence $\{(x_n, x_n)\}$ $\tilde{g}α$-converges to $(x, x)$ which clearly belongs to $\Delta$. Therefore, $\Delta$ is a sequentially $\tilde{g}α$-closed subset of $X \times X$. Conversely, suppose that the diagonal $\Delta$ is a sequentially $\tilde{g}α$-closed subset of $X \times X$. Assume that a sequence $\{x_n\}$ is $\tilde{g}α$-converging to $x$ and $y$. Then it follows that $\{(x_n, x_n)\}$ $\tilde{g}α$-converges to $(x, y)$. By hypothesis, since $\Delta$ is sequentially $\tilde{g}α$-closed, we have $(x, y) \in \Delta$. Thus $x = y$. Therefore, $X$ is $\tilde{g}α-US$. □

**Theorem 2.8.** If a space $X$ is $\tilde{g}α-US$ and a subset $M$ of $X$ is sequentially $\tilde{G}αO$-compact, then $M$ is sequentially $\tilde{g}α$-closed.

**Proof.** Assume that $\{x_n\}$ is any sequence in $M$ which $\tilde{g}α$-converges to a point $x \in X$. Since $M$ is sequentially $\tilde{G}αO$-compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ $\tilde{g}α$-converges to $m \in M$. Since $X$ is $\tilde{g}α-US$, we have $x = m$. This shows that $M$ is sequentially $\tilde{g}α$-closed. □

**Theorem 2.9.** The product space of an arbitrary family of $\tilde{g}α-US$ topological spaces is a $\tilde{g}α-US$ topological space.

**Proof.** Let $\{X_\lambda : \lambda \in \Delta\}$ be a family of $\tilde{g}α-US$ topological spaces with the index set $\Delta$. The product space of $\{X_\lambda : \lambda \in \Delta\}$ is denoted by $\prod X_\lambda$. Let $\{x_n(\lambda)\}$ be a sequence in $\prod X_\lambda$. Suppose that $\{x_n(\lambda)\}$ $\tilde{g}α$-converges to two distinct points $x$ and $y$ in $\prod X_\lambda$. Then there exists a $\lambda_0 \in \Delta$ such that $x(\lambda_0) \neq y(\lambda_0)$. Then $\{x_n(\lambda_0)\}$ is a sequence in $X_{\lambda_0}$. Let $V_{\lambda_0}$ be any $\tilde{g}α$-open in $X_{\lambda_0}$ containing $x(\lambda_0)$. Then $V = V_{\lambda_0} \times \prod_{\lambda \neq \lambda_0} X_\lambda$ is a $\tilde{g}α$-open set of $\prod X_\lambda$ containing $x$. Therefore, $\{x_n(\lambda)\}$ is eventually in $V$. Thus $\{x_n(\lambda_0)\}$ is eventually in $V_{\lambda_0}$ and it $\tilde{g}α$-converges to $x(\lambda_0)$. Similarly, the sequence $\{x_n(\lambda_0)\}$ $\tilde{g}α$-converges to $y(\lambda_0)$. This is a contradiction as $X_{\lambda_0}$ is a $\tilde{g}α-US$ space. Therefore, the product space $\prod X_\lambda$ is $\tilde{g}α-US$. □

3. Sequentially $\tilde{G}αO$-compact preserving functions

**Definition 3.1.** A function $f : X \to Y$ is said to be
(i) Sequentially \( \tilde{g}_\alpha \)-continuous at \( x \in X \) if the sequence \( \{ f(x_n) \} \) \( \tilde{g}_\alpha \)-converges to \( f(x) \) whenever a sequence \( \{ x_n \} \) \( \tilde{g}_\alpha \)-converges to \( x \). If \( f \) is sequentially \( \tilde{g}_\alpha \)-continuous at each \( x \in X \), then it is said to be sequentially \( \tilde{g}_\alpha \)-continuous.

(ii) Sequentially nearly \( \tilde{g}_\alpha \)-continuous, if for each sequence \( \{ x_n \} \) in \( X \) that \( \tilde{g}_\alpha \)-converges to \( x \in X \), there exists subsequence \( \{ x_{n_k} \} \) of \( \{ x_n \} \) such that the sequence \( \{ f(x_{n_k}) \} \) \( \tilde{g}_\alpha \)-converges to \( f(x) \).

(iii) Sequentially sub \( \tilde{g}_\alpha \)-continuous if for each point \( x \in X \) and each sequence \( \{ x_n \} \) in \( X \) \( \tilde{g}_\alpha \)-converging to \( x \), there exists a subsequence \( \{ x_{n_k} \} \) of \( \{ x_n \} \) and a point \( y \in Y \) such that the sequence \( \{ f(x_{n_k}) \} \) \( \tilde{g}_\alpha \)-converges to \( y \).

(iv) Sequentially \( \tilde{G}_\alpha O \)-compact preserving if the image \( f(M) \) of every sequentially \( \tilde{G}_\alpha O \)-compact set \( M \) of \( X \) is a sequentially \( \tilde{G}_\alpha O \)-compact subset of \( Y \).

**Theorem 3.2.** Let \( f_1 : X \rightarrow Y \) and \( f_2 : X \rightarrow Y \) be two sequentially \( \tilde{g}_\alpha \)-continuous functions. If \( Y \) is \( \tilde{g}_\alpha \)-US, then the set \( E = \{ x \in X : f_1(x) = f_2(x) \} \) is sequentially \( \tilde{g}_\alpha \)-closed.

**Proof.** Suppose that \( Y \) is \( \tilde{g}_\alpha \)-US and \( \{ x_n \} \) is any sequence in \( E \) that \( f_1 \)-converges to \( x \in X \). Since \( f_1 \) and \( f_2 \) are sequentially \( \tilde{g}_\alpha \)-continuous functions, the sequence \( \{ f_1(x_n) \} \) (respectively, \( \{ f_2(x_n) \} \)) converges to \( f_1(x) \) (respectively, \( f_2(x) \)). Since \( x_n \in E \) for each \( n \in N \) and \( Y \) is \( \tilde{g}_\alpha \)-US, \( f_1(x) = f_2(x) \) and hence \( x \in E \). This shows that \( E \) is sequentially \( \tilde{g}_\alpha \)-closed. \( \square \)

**Lemma 3.3.** Every function \( f : X \rightarrow Y \) is sequentially sub \( \tilde{g}_\alpha \)-continuous if \( Y \) is sequentially \( \tilde{G}_\alpha O \)-compact.

**Proof.** Let \( \{ x_n \} \) be a sequence in \( X \) that \( \tilde{g}_\alpha \)-converges to \( x \in X \). It follows that \( \{ f(x_n) \} \) is a sequence in \( Y \). Since \( Y \) is sequentially \( \tilde{G}_\alpha O \)-compact, there exists a subsequence \( \{ f(x_{n_k}) \} \) of \( \{ f(x_n) \} \) that \( \tilde{g}_\alpha \)-converges to a point \( y \in Y \). Therefore \( f : X \rightarrow Y \) is sequentially sub \( \tilde{g}_\alpha \)-continuous. \( \square \)

**Theorem 3.4.** Every sequentially nearly \( \tilde{g}_\alpha \)-continuous function is sequentially \( \tilde{G}_\alpha O \)-compact preserving.

**Proof.** Let \( f : X \rightarrow Y \) be a sequentially nearly \( \tilde{g}_\alpha \)-continuous function and \( M \) be any sequentially \( \tilde{G}_\alpha O \)-compact subset of \( X \). We will show that \( f(M) \) is a sequentially \( \tilde{G}_\alpha O \)-compact subset of \( Y \). So, assume that \( \{ y_n \} \) is any sequence in \( f(M) \). Then for each \( n \in N \), there exists a point \( x_n \in M \) such that \( f(x_n) = y_n \). Now \( M \) is sequentially \( \tilde{G}_\alpha O \)-compact, so
there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) that \( \tilde{\gamma}a \)-converges to a point \( x \in M \). Since \( f \) is sequentially nearly \( \tilde{\gamma}a \)-continuous, there exist a subsequence \( \{x_{n_k}(i)\} \) of \( \{x_{n_k}\} \) such that \( \{f(x_{n_k}(i))\} \) \( \tilde{\gamma}a \)-converges to \( f(x) \). Therefore, there exists a subsequence \( \{y_{n_k}(i)\} \) of \( \{y_{n_k}\} \) that \( \tilde{\gamma}a \)-converges to \( f(x) \). This implies that \( f(M) \) is a sequentially \( \tilde{\gamma}aO \)-compact set of \( Y \).

**Theorem 3.5.** Every sequentially \( \tilde{\gamma}aO \)-compact preserving function is sequentially sub \( \tilde{\gamma}a \)-continuous.

**Proof.** Suppose that \( f : X \to Y \) is a sequentially \( \tilde{\gamma}aO \)-compact preserving function. Let \( x \) be any point of \( X \) and \( \{x_n\} \) a sequence that \( \tilde{\gamma}a \)-converges to \( x \). We denote the set \( \{x_n : n \in N\} \) by \( A \) and put \( M = A \cup \{x\} \). Since \( \{x_n\} \) \( \tilde{\gamma}a \)-converges to \( x \), \( M \) is sequentially \( \tilde{\gamma}aO \)-compact. By hypothesis, \( f \) is sequentially \( \tilde{\gamma}aO \)-compact subset of \( Y \). Now in \( f(M) \) there exists a subsequence \( \{f(x_{n_k})\} \) of \( \{f(x_n)\} \) that \( \tilde{\gamma}a \)-converges to a point \( y \in f(M) \). This implies that \( f \) sequentially sub \( \tilde{\gamma}a \)-continuous.

**Theorem 3.6.** A function \( f : X \to Y \) is sequentially \( \tilde{\gamma}aO \)-compact preserving if and only if \( f/M : M \to f(M) \) is sequentially sub \( \tilde{\gamma}a \)-continuous for each sequentially \( \tilde{\gamma}aO \)-compact set \( M \) of \( X \).

**Proof.**

**Necessity:** Suppose that \( f : X \to Y \) is a sequentially \( \tilde{\gamma}aO \)-compact preserving function. Then \( f(M) \) is sequentially \( \tilde{\gamma}aO \)-compact in \( Y \) for each sequentially \( \tilde{\gamma}aO \)-compact subset \( M \) of \( X \). Therefore, by Theorem 3.5 \( f/M : M \to f(M) \) is sequentially sub \( \tilde{\gamma}a \)-continuous.

**Sufficiency:** Let \( M \) be any sequentially \( \tilde{\gamma}aO \)-compact set of \( X \). We will show that \( f(M) \) is sequentially \( \tilde{\gamma}aO \)-compact subset of \( Y \). Let \( \{y_n\} \) be any sequence in \( f(M) \). Then for each \( n \in N \), there exists a point \( x_n \in M \) such that \( f(x_n) = y_n \). Since \( \{x_n\} \) is a sequence in the sequentially \( \tilde{\gamma}aO \)-compact set \( M \) there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) that \( \tilde{\gamma}a \)-converges to a point in \( M \). By hypothesis \( f/M : M \to f(M) \) is sequentially sub \( \tilde{\gamma}a \)-continuous, hence there exists a subsequence \( \{y_{n_k}\} \) of \( \{y_n\} \) that \( \tilde{\gamma}a \)-converges to \( y \in f(M) \). This implies that \( f(M) \) is sequentially \( \tilde{\gamma}aO \)-compact in \( Y \).

**Corollary 3.7.** If a function \( f : X \to Y \) is sequentially sub \( \tilde{\gamma}a \)-continuous and \( f(M) \) is sequentially \( \tilde{\gamma}a \)-closed in \( Y \) for each sequentially \( \tilde{\gamma}aO \)-compact set \( M \) of \( X \), then \( f \) is sequentially \( \tilde{\gamma}aO \)-compact preserving.
Proof. It will sufficient to show that \( f/M : M \to f(M) \) is sequentially sub \( \tilde{\alpha} \)-continuous for each sequentially \( \tilde{\alpha} \)-compact set \( M \) of \( X \) and by Lemma 3.3. we are done. So, let \( \{x_n\} \) be any sequence in \( M \) that \( \tilde{\alpha} \)-converges to a point \( x \in M \). Then, since \( f \) is sequentially sub \( \tilde{\alpha} \)-continuous there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) and a point \( y \in Y \) such that \( f(x_{n_k}) \to y \). Since \( \{f(x_{n_k})\} \) is a sequence in the sequentially \( \tilde{\alpha} \)-closed set \( f(M) \) of \( Y \), we obtain \( y \in f(M) \). This implies that \( f/M : M \to f(M) \) is sequentially sub \( \tilde{\alpha} \)-continuous. \( \Box \)

4. Slightly \( \tilde{\alpha} \)-Continuous Functions

**Definition 4.1.** A function \( f : X \to Y \) is said to be slightly \( \tilde{\alpha} \)-continuous if for each \( x \in X \) and for each \( v \in CO(Y, f(x)) \), there exists \( U \in \tilde{\alpha}O(X, x) \) such that \( f(U) \subset V \), where \( CO(Y, f(x)) \) is the family of clopen sets containing \( f(x) \) in a space \( Y \).

**Definition 4.2.** Let \((D, \leq)\) be a directed set \( A \) net \( \{x_\lambda : \lambda \in D\} \) in \( X \) is said to be \( \tilde{\alpha} \)-convergent to a point \( x \in X \) if \( \{x_\lambda\}_{\lambda \in D} \) is eventually in each \( V \in \tilde{\alpha}O(X, x) \).

**Theorem 4.3.** For a function \( f : X \to Y \), the following are equivalent:

(a) \( f \) is slightly \( \tilde{\alpha} \)-continuous.
(b) \( f^{-1}(v) \in \tilde{\alpha}O(X) \) for each \( V \in CO(Y) \).
(c) \( f^{-1}(v) \) is \( \tilde{\alpha} \)-clopen for each \( V \in CO(Y) \).
(d) for each \( x \in X \) and for each net \( \{x_\lambda\}_{\lambda \in D} \) in \( X \).

Proof. (a) \( \Rightarrow \) (b). Let \( V \in CO(Y) \) and let \( x \in f^{-1}(V) \). Then \( f(x) \in V \). Since \( f \) is slightly \( \tilde{\alpha} \)-continuous, there is an \( U \in \tilde{\alpha}O(X, x) \) such that \( f(U) \subset V \). Thus \( f^{-1}(U) = \bigcup \{U : x \in f^{-1}(V)\} \), that is \( f^{-1}(U) \) is a union of \( \tilde{\alpha} \)-open sets. Hence \( f^{-1}(U) \in \tilde{\alpha}O(X) \).

(b) \( \Rightarrow \) (c). Let \( V \in CO(Y) \). Then \((Y - V) \in CO(X) \). By hypothesis \( f^{-1}(Y - V) = X - f^{-1}(V) \in \tilde{\alpha}O(X) \). Thus \( f^{-1}(V) \) is \( \tilde{\alpha} \)-closed.

(c) \( \Rightarrow \) (d). Let \( \{x_\lambda\}_{\lambda \in D} \) be a net in \( X \) \( \tilde{\alpha} \)-converging to \( x \) and let \( V \in CO(Y, f(x)) \) be a net. There is thus a \( U \in \tilde{\alpha}O(X, x) \) such that \( f(U) \subset V \). There is thus a \( \lambda_0 \in D \) such that \( \lambda_0 \leq \lambda \) implies \( x_\lambda \in U \) and \( \{x_\lambda\}_{\lambda \in D} \) is \( \tilde{\alpha} \)-convergent to \( x \). Thus \( f(x_\lambda) \in f(U) \subset V \) for all \( \lambda \). Thus \( \{f(x_\lambda)\}_{\lambda \in D} \) is \( \tilde{\alpha} \)-convergent to \( f(x) \).

(d) \( \Rightarrow \) (a). Suppose that \( f \) is not slightly \( \tilde{\alpha} \)-continuous at a point \( x \in X \), then there exists a \( V \in CO(Y, f(x)) \) such that \( f(U) \) does not contained in \( V \) for each \( U \in \tilde{\alpha}O(X, x) \). So \( f(U) \cap (Y - V) \neq \emptyset \) and thus
If $f$ is continuous, then for each $y \in Y$, $f^{-1}(y) \neq \emptyset$. Thus, for each $y \in Y$, there exists a selection function $x_U$ from $\tilde{\gamma}O(X, x)$ into $X$ for each $U \in \tilde{\gamma}O(X, x)$, and $\{x_U\} \in \tilde{\gamma}O(X, x)$ is an infinite net in $X$ $\tilde{\gamma}$-converging to $x$. Since $x_U \in U \cap f^{-1}(y - V) = U - f^{-1}(V)$ and so $f(x_U) \notin V$, for each $U$, $\{f(x_U)\} \in \tilde{\gamma}O(X, x)$ is not eventually in $V \cap CO(Y, f(x))$, which is a contradiction. Hence (a) holds.

**Theorem 4.4.** If $f : X \to Y$ is slightly $\tilde{\gamma}$-continuous and $g : Y \to Z$ is slightly continuous, then their composition $g \circ f$ is slightly $\tilde{\gamma}$-continuous.

**Proof.** Let $V \in CO(Z)$, then $g^{-1}(V) \in CO(Y)$ [6]. Since $f$ is slightly $\tilde{\gamma}$-continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \in \tilde{\gamma}O(X)$. Thus $g \circ f$ is slightly $\tilde{\gamma}$-continuous.

**Theorem 4.5.** The following are equivalent for a function $f : X \to Y$:

(a) $f$ is slightly $\tilde{\gamma}$-continuous,
(b) for each $x \in X$ and for each $V \in CO(Y, f(x))$, there exists a $\tilde{\gamma}$-closed set $U$ such that $f(U) \subset U$,
(c) for each closed set $F$ of $Y$, $f^{-1}(F)$ is $\tilde{\gamma}$-closed,
(d) $f(cl(A)) \subset \tilde{\gamma}ocl(f(A))$ for each $A \subset X$ and
(e) $cl(f^{-1}(B)) \subset f^{-1}(\tilde{\gamma}ocl(B))$ for each $B \subset Y$.

**Proof.** (a) $\Rightarrow$ (b) Let $x \in X$ and $V \in CO(Y, f(x))$ by Theorem 4.3. $f^{-1}(V)$ is clopen. Put $U = f^{-1}(V)$, then $x \in U$ and $f(U) \subset V$.

(b) $\Rightarrow$ (c) It is obvious.

(c) $\Rightarrow$ (d) Since $\tilde{\gamma}ocl(f(A))$ is the smallest $\tilde{\gamma}$-closed set containing $f(A)$, hence by (c), we have (d).

(d) $\Rightarrow$ (e) For each $B \subset Y$, $f(cl(f^{-1}(B))) \subset \tilde{\gamma}ocl(f(f^{-1}(B))) \subset \tilde{\gamma}ocl(B)$. Hence $f(cl(f^{-1}(B))) \subset \tilde{\gamma}ocl(B) \Rightarrow cl(f^{-1}(B)) \subset f^{-1}(\tilde{\gamma}ocl(B))$.

(e) $\Rightarrow$ (a) Let $V \in CO(Y)$. Then $(Y - V) \in CO(X)$, by (e), we have $cl(f^{-1}(Y - V)) \subset f^{-1}(\tilde{\gamma}ocl(Y - V)) = f^{-1}(Y - V)$, since every closed set is $\tilde{\gamma}$-closed, thus $f^{-1}(Y - V) = X - f^{-1}(V)$ is closed and thus $\tilde{\gamma}$-closed, thus $f^{-1}(V) \in \tilde{\gamma}O(X)$ and $f$ is slightly $\tilde{\gamma}$-continuous.

**Theorem 4.6.** If $f : X \to Y$ is a slightly $\tilde{\gamma}$-continuous injection and $Y$ is clopen $T_1$, then $X$ is $\tilde{\gamma}$-$T_1$.

**Proof.** Suppose that $Y$ is clopen $T_1$. For any distinct points $x$ and $y$ in $X$, there exist $V, W \in CO(Y)$ such that $f(x) \in V$, $f(y) \notin V$, $f(x) \notin W$ and $f(y) \in W$. Since $f$ is slightly $\tilde{\gamma}$-continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are $\tilde{\gamma}$-open subsets of $X$ such that $x \notin f^{-1}(V)$, $y \notin f^{-1}(V)$, $x \notin f^{-1}(W)$ and $y \in f^{-1}(W)$. This shows that $X$ is $\tilde{\gamma}$-$T_1$.  

\[ \square\]
Theorem 4.7. If \( f : X \to Y \) is a slightly \( \tilde{g}_\alpha \)-continuous surjection and \( Y \) is clopen \( T_2 \), then \( X \) is \( \tilde{g}_\alpha-T_2 \).

Proof. For any pair of distinct points \( x \) and \( y \) in \( X \), there exist disjoint clopen sets \( U \) and \( V \) in \( Y \) such that \( f(x) \in U \) and \( f(y) \in V \). Since \( f \) is slightly \( \tilde{g}_\alpha \)-continuous, \( f^{-1}(U) \) and \( f^{-1}(V) \) are \( \tilde{g}_\alpha \)-open in \( X \) containing \( x \) and \( y \) respectively. Therefore \( f^{-1}(U) \cap f^{-1}(V) = \emptyset \) because \( U \cap V = \emptyset \). This shows that \( X \) is \( \tilde{g}_\alpha-T_2 \). \( \square \)

Definition 4.8. A space is called \( \tilde{g}_\alpha \)-regular if for each \( \tilde{g}_\alpha \)-closed set \( F \) and each point \( x \notin F \), there exist disjoint open sets \( U \) and \( V \) such that \( F \subset U \) and \( x \in V \).

Definition 4.9. A space is said to be \( \tilde{g}_\alpha \)-normal if for every pair of disjoint \( \tilde{g}_\alpha \)-closed subsets \( F_1 \) and \( F_2 \) of \( X \), there exist disjoint open sets \( U \) and \( V \) such that \( F_1 \subset U \) and \( F_2 \subset V \).

Theorem 4.10. If \( f \) is slightly \( \tilde{g}_\alpha \)-continuous injective open function from a \( \tilde{g}_\alpha \)-regular space \( X \) onto a space \( Y \), then \( Y \) is clopen regular.

Proof. Let \( F \) be clopen set in \( Y \) and be \( y \notin F \). Take \( y = f(x) \). Since \( f \) is slightly \( \tilde{g}_\alpha \)-continuous, \( f^{-1}(F) \) is a \( \tilde{g}_\alpha \)-closed set. Take \( G = f^{-1}(F) \), we have \( x \notin G \). Since \( X \) is \( \tilde{g}_\alpha \)-regular, there exist disjoint open sets \( U \) and \( V \) such that \( G \subset U \) and \( x \in V \). We obtain that \( F = f(G) \subset f(U) \) and \( y = f(x) \in f(V) \) such that \( f(U) \) and \( f(V) \) are disjoint open sets. This shows that \( Y \) is clopen regular. \( \square \)

Theorem 4.11. If \( f \) is slightly \( \tilde{g}_\alpha \)-continuous injective open function from a \( \tilde{g}_\alpha \)-normal space \( X \) onto a space \( Y \), then \( Y \) is clopen normal.

Proof. Let \( F_1 \) and \( F_2 \) be disjoint clopen subsets of \( Y \). Since \( f \) is slightly \( \tilde{g}_\alpha \)-continuous, \( f^{-1}(F_1) \) and \( f^{-1}(F_2) \) are \( \tilde{g}_\alpha \)-closed sets. Take \( U = f^{-1}(F_1) \) and \( V = f^{-1}(F_2) \). We have \( U \cap V = \emptyset \). Since \( X \) is \( \tilde{g}_\alpha \)-regular, there exist disjoint open sets \( A \) and \( B \) such that \( U \subset A \) and \( V \subset B \). We obtain that \( F_1 = f(U) \subset f(A) \) and \( F_2 = f(V) \subset f(B) \) such that \( f(A) \) and \( f(B) \) are disjoint open sets. Thus, \( Y \) is clopen normal. \( \square \)

References

[2] R. Devi and A. Selvakumar, On \( \tilde{g}_\alpha \)-closed \( \text{sets} \) in \( \text{Topological spaces} \) (Submitted).
Application of $\Theta_{n}$-closed sets


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