SUBMANIFOLDS OF AN ALMOST $\tau$-PARACONTACT RIEMANNIAN MANIFOLD ENDOWED WITH A QUARTER-SYMMETRIC NON-METRIC CONNECTION

MOBIN AHMAD*, JAE-BOK JUN**, AND ABDUL HASEEB***

Abstract. We define a quarter-symmetric non-metric connection in an almost $\tau$-paracontact Riemannian manifold and we consider the submanifolds of an almost $\tau$-paracontact Riemannian manifold endowed with a quarter-symmetric non-metric connection. We also obtain the Gauss, Codazzi and Weingarten equations and the curvature tensor for the submanifolds of an almost $\tau$-paracontact Riemannian manifold endowed with a quarter-symmetric non-metric connection.

1. Introduction

In [9], R. S. Mishra studied almost complex and almost contact submanifolds. And in [3], S. Ali and R. Nivas considered submanifolds of a Riemannian manifold with a quarter-symmetric connection. Some properties of submanifolds of a Riemannian manifold with a quarter-symmetric semi-metric connection were studied in [6] by L. S. Dass etc. Moreover, in [8], I. Mihai and K. Matsumoto studied the submanifolds of an almost $\tau$-paracontact Riemannian manifold of $P$-Sasakian type.

Let $\nabla$ be a linear connection in an $n$-dimensional differentiable manifold $M$. The torsion tensor $T$ and the curvature tensor $R$ of $\nabla$ are given respectively by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$
The connection $\nabla$ is symmetric if its torsion tensor $T$ vanishes, otherwise it is non-symmetric. The connection $\nabla$ is metric connection if there is a Riemannian metric $g$ in $M$ such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is Levi-Civita connection.

In [7], S. Golab introduced the idea of a quarter-symmetric linear connection if its torsion tensor $T$ is of the form

$$T(X;Y) = u(Y)X - u(X)Y,$$

where $u$ is a 1-form and $\psi$ is a tensor field of type (1,1). In [10], R. S. Mishra and S. N. Pandey considered a quarter-symmetric metric connection and studied some of its properties. In [1], [2], [4], [11], [12] and [13], some kinds of quarter-symmetric metric connections were studied.

Let $M$ be an $n$-dimensional Riemannian manifold with a positive definite metric $g$. If there exist a tensor field $\psi$ of type (1,1), $r$-vector fields $\xi_1, \xi_2, ..., \xi_r$ ($n > r$), $r$ 1-forms $\eta^1, \eta^2, ..., \eta^r$ such that

(i) $\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta$, $\alpha, \beta \in \{1, 2, ..., r\}$,
(ii) $\psi^\alpha(X) = X - \eta^\alpha(X)\xi_\alpha$,
(iii) $\eta^\alpha(X) = g(X, \xi_\alpha)$, $\alpha \in \{r\}$,
(iv) $g(\psi X, \psi Y) = g(X, Y) - \Sigma_\alpha \eta^\alpha(X)\eta^\alpha(Y),$

where $X$ and $Y$ are vector fields on $M$ and $a^\alpha b_\alpha \overset{\text{def}}{=} \Sigma_\alpha a^\alpha b_\alpha$, then the structure $\Sigma = (\psi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in \{r\}}$ is said to be an almost $r$-paracontact Riemannian structure on $M$ and $M$ is an almost $r$-paracontact Riemannian manifold [1].

With the help of the above conditions (i), (ii), (iii) and (iv) we have (v) $\psi(\xi_\alpha) = 0$, $\alpha \in \{r\}$,
(vi) $\eta^a \circ \psi = 0$, $\alpha \in \{r\}$,
(vii) $\Psi(X, Y) \overset{\text{def}}{=} g(\psi X, Y) = g(X, \psi Y)$.

An almost $r$-paracontact Riemannian manifold $M$ with a structure $\Sigma = (\psi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in \{r\}}$ is said to be of $S$-paracontact type [1] if

$$\Psi(X, Y) = (\nabla^*_X \eta^\alpha)(X), \quad \alpha \in \{r\}. $$

An almost $r$-paracontact Riemannian manifold $M$ with a structure $\Sigma = (\psi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in \{r\}}$ is said to be of $P$-Sasakian type if it also satisfies

$$(\nabla^*_Z \Psi)(X, Y) = -\Sigma_\alpha \eta^\alpha(X)[g(Y, Z) - \Sigma_\beta \eta^\beta(Y)\eta^\beta(Z)]$$

for all vector fields $X$, $Y$ and $Z$ on $M$ [8].
The conditions given as above are equivalent respectively to
\[ \psi X = \nabla_X \xi_\alpha, \quad \alpha \in (r) \]
and
\[ (\nabla^*_\psi)(X) = -\Sigma_\alpha \eta^\alpha (X)[Y - \eta^\alpha (Y) \xi_\alpha] - [g(X, Y) - \Sigma_\alpha \eta^\alpha (X) \eta^\alpha (Y)] \Sigma_\beta \xi_\beta. \]

In this paper, we study quarter-symmetric non-metric connection in an almost \( r \)-paracontact Riemannian manifold. We consider the hypersurfaces and submanifolds of an almost \( r \)-paracontact Riemannian manifold endowed with a quarter-symmetric non-metric connection. We also obtain the Gauss and Codazzi equations for hypersurfaces, curvature tensor and the Weingarten equation for submanifolds of an almost \( r \)-paracontact Riemannian manifold with respect to the quarter-symmetric non-metric connection.

2. Preliminaries

Let \( M^{n+1} \) be an \((n+1)\)-dimensional differentiable manifold of class \( C^\infty \) and let \( M^n \) be the hypersurface in \( M^{n+1} \) by the immersion \( \tau : M^n \to M^{n+1} \). The differential \( d\tau \) of the immersion \( \tau \) is denoted by \( B \). The vector field \( X \) in the tangent space of \( M^n \) corresponds to a vector field \( BX \) in that of \( M^{n+1} \). Suppose that the enveloping manifold \( M^{n+1} \) is an almost \( r \)-paracontact Riemannian manifold with metric \( \tilde{g} \). Then the hypersurface \( M^n \) is also an almost \( r \)-paracontact Riemannian manifold with the induced metric \( g \) defined by
\[ g(\psi X, Y) = \tilde{g}(B \psi X, BY), \]
where \( X \) and \( Y \) are arbitrary vector fields and \( \psi \) is a tensor of type \((1,1)\) on \( M^n \). If the Riemannian manifolds \( M^{n+1} \) and \( M^n \) are both orientable, we can choose a unique vector field \( N \) defined along \( M^n \) such that
\[ \tilde{g}(B \psi X, N) = 0 \quad \text{and} \quad \tilde{g}(N, N) = 1 \]
for arbitrary vector field \( X \) in \( M^n \). We call this vector field as a normal vector field to the hypersurface \( M^n \).

Now, we define a quarter-symmetric non-metric connection \( \tilde{\nabla} \) by ([1], [2])
\[ \tilde{\nabla}_X \tilde{Y} = \tilde{\nabla}_X \tilde{Y} + \tilde{\eta}^\alpha (\tilde{Y}) \tilde{\psi} \tilde{X} \tag{2.1} \]
for arbitrary vector fields \( \tilde{X} \) and \( \tilde{Y} \) tangents to \( M^{n+1} \), where \( \tilde{\nabla} \) denotes the Levi-Civita connection with respect to the Riemannian metric \( \tilde{g}, \tilde{\eta}^\alpha \).
is a 1-form, \( \tilde{\xi}_\alpha \) is the vector field defined by
\[
\tilde{g}(\tilde{\xi}_\alpha, \tilde{X}) = \tilde{\eta}^\alpha(\tilde{X})
\]
for an arbitrary vector field \( \tilde{X} \) of \( M^{n+1} \). Also
\[
\tilde{g}(\tilde{\psi} \tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{\psi} \tilde{Y}),
\]
where \( \tilde{\psi} \) is a tensor of type \((1,1)\).

Now, suppose that \( \Sigma = (\tilde{\psi}, \tilde{\xi}_\alpha, \tilde{\eta}^\alpha, \tilde{g})_{\alpha \in (r)} \) is an almost \( r \)-paracontact Riemannian structure on \( M^{n+1} \), then every vector field \( \tilde{X} \) on \( M^{n+1} \) is decomposed as
\[
\tilde{X} = B X + \lambda(X) N,
\]
where \( \lambda \) is a 1-form on \( M^{n+1} \) and for any vector field \( X \) on \( M^n \) and normal \( N \). Also we have \( b(BX) = b(X), \psi BX = B \psi X \) and \( \eta^\alpha(BX) = \eta^\alpha(X) \), where \( b \) is a 1-form on \( M^n \).

For each \( \alpha \in (r) \), we have [2]
\[
\begin{align*}
(2.2) \quad \tilde{\psi} BX &= B\psi X + b(X) N \quad \text{and} \quad \psi N = B N' + KN, \\
(2.3) \quad a_\alpha &= \eta^\alpha(N), \quad \alpha \in (r).
\end{align*}
\]

Now, we define \( \tilde{\eta}^\alpha \) as
\[
(2.4) \quad \tilde{\eta}^\alpha(BX) = \eta^\alpha(X), \quad \alpha \in (r).
\]

**Theorem 2.1.** The connection induced on the hypersurface of a Riemannian manifold with a quarter-symmetric non-metric connection with respect to the unit normal vector is also a quarter-symmetric non-metric connection.

**Proof.** Let \( \tilde{\nabla} \) be the induced connection from \( \tilde{\nabla} \) on the hypersurface with respect to the unit normal vector \( N \), then we have
\[
(2.5) \quad \tilde{\nabla}_{BX} BY = B(\tilde{\nabla}_X Y) + h(X, Y) N
\]
for arbitrary vector fields \( X \) and \( Y \) on \( M^n \), where \( h \) is the second fundamental tensor of the hypersurface \( M^n \). Let \( \nabla \) be the connection induced on the hypersurface from \( \nabla \) with respect to the unit normal vector \( N \), then we have
\[
(2.6) \quad \tilde{\nabla}_{BX} BY = B(\nabla_X Y) + m(X, Y) N
\]
for arbitrary vector fields \( X \) and \( Y \) of \( M^n \), \( m \) being a tensor field of type \((0,2)\) on the hypersurface \( M^n \).
From equation (2.1), we have
\[ \tilde{\nabla}_{BX}BY = \tilde{\nabla}_{BX}BY + \tilde{\eta}^a(BY)\tilde{\psi}BX. \]
Using (2.2), (2.4), (2.5) and (2.6) in the above equation, we get
\[ B(\nabla_XY) + m(X,Y)N = B(\tilde{\nabla}_XY) + h(X,Y)N + \eta^a(Y)B\psi X + \eta^a(Y)b(X)N. \]
Comparison of the tangential and normal parts in the above equation yield
\[ \nabla_XY = \tilde{\nabla}_XY + \eta^a(Y)\psi X \]
and
\[ m(X,Y) = h(X,Y) + \eta^a(Y)b(X). \]
Thus we have
\[ \nabla_XY - \nabla_YX - [X,Y] = \eta^a(Y)\psi X - \eta^a(X)\psi Y. \]
Hence the connection \( \nabla \) induced on \( M^n \) is a quarter-symmetric non-metric connection [7].

3. Totally geodesic and totally umbilical hypersurfaces

We define \( \tilde{\nabla}B \) and \( \nabla B \) respectively by
\[ (\tilde{\nabla}B)(X,Y) = (\tilde{\nabla}_XB)(Y) = \tilde{\nabla}_{BX}BY - B(\tilde{\nabla}_XY) \]
and
\[ (\nabla B)(X,Y) = (\nabla_XB)(Y) = \nabla_{BX}BY - B(\nabla_XY), \]
where \( X \) and \( Y \) are arbitrary vector fields on \( M^n \). Then (2.5) and (2.6) take the form respectively
\[ (\tilde{\nabla}_XB)Y = h(X,Y)N \]
and
\[ (\nabla_XB)Y = m(X,Y)N. \]
These are the Gauss equations with respect to the induced connection \( \tilde{\nabla} \) and \( \nabla \), respectively.

Let \( X_1, X_2, ..., X_n \) be \( n \)-orthonormal vector fields. Then the function
\[ \frac{1}{n} \sum_{i=1}^{n} h(X_i, X_i) \]
is called the mean curvature of $M^n$ with respect to the Riemannian connection $\nabla$ and
\[
\frac{1}{n} \sum_{i=1}^{n} m(X_i, X_i)
\]
is called the mean curvature of $M^n$ with respect to the quarter-symmetric non-metric connection $\nabla$.

From this we have following definitions:

**Definition 3.1.** The hypersurface $M^n$ is called totally geodesic of $M^{n+1}$ with respect to the Riemannian connection $\nabla$ if $h$ vanishes.

**Definition 3.2.** The hypersurface $M^n$ is called totally umbilical with respect to the connection $\nabla$ if $h$ is proportional to the metric tensor $g$.

We call $M^n$ is totally geodesic and totally umbilical with respect to the quarter-symmetric non-metric connection $\nabla$ according as the function $m$ vanishes and proportional to the metric $g$, respectively.

Now we have the following theorems:

**Theorem 3.3.** In order that the mean curvature of the hypersurface $M^n$ with respect to the Riemannian connection $\nabla$ coincides with that of $M^n$ with respect to the quarter-symmetric non-metric connection $\nabla$ if and only if $M^n$ is invariant.

**Proof.** In view of (2.7), we have
\[
m(X_i, X_i) = h(X_i, X_i) + \eta^a(Y_i)b(X_i).
\]
Summing up for $i = 1, 2, ..., n$ and divide by $n$, we obtain
\[
\frac{1}{n} \sum_{i=1}^{n} m(X_i, X_i) = \frac{1}{n} \sum_{i=1}^{n} h(X_i, X_i)
\]
if and only if $b(X_i) = 0$, which gives the proof of our theorem.

**Theorem 3.4.** The hypersurface $M^n$ is totally geodesic with respect to the Riemannian connection $\nabla$ if and only if it is totally geodesic with respect to the quarter-symmetric non-metric connection $\nabla$, provided that $M^n$ is invariant.

**Proof.** The proof follows from (2.7) easily.
4. Gauss, Weingarten and Codazzi equations

In this section we shall obtain the Weingarten equation with respect to the quarter-symmetric non-metric connection \( \tilde{\nabla} \). For the Riemannian connection \( \nabla \), these equations are given by

\[
(4.1) \quad \tilde{\nabla}_X N = -BHX
\]

for any vector field \( X \) in \( M^n \), where \( H \) is a tensor field of type (1,1) of \( M^n \) defined by

\[
g(HX, Y) = h(X, Y)
\]

from equations (2.1), (2.2) and (2.3) we have

\[
\tilde{\nabla}_X N = \tilde{\nabla}_X N + a_\alpha [B(\psi X) + b(X)N].
\]

Using (4.1) we have

\[
(4.2) \quad \tilde{\nabla}_X N = -BMX + a_\alpha b(X)N,
\]

where \( M = H - a_\alpha \psi \), and \( X \) is any vector field in \( M^n \).

Equation (4.2) is the Weingarten equation with respect to the quarter-symmetric non-metric connection.

We shall find the equations of Gauss and Codazzi with respect to the quarter-symmetric non-metric connection. The curvature tensor with respect to the quarter-symmetric non-metric connection \( \tilde{\nabla} \) of \( M^{n+1} \) is

\[
\tilde{R}(\tilde{X}, \tilde{Y}) \tilde{Z} = \tilde{\nabla}_\tilde{X} \tilde{\nabla}_\tilde{Y} \tilde{Z} - \tilde{\nabla}_\tilde{Y} \tilde{\nabla}_\tilde{X} \tilde{Z} - \tilde{\nabla}_{[\tilde{X}, \tilde{Y}]} \tilde{Z}.
\]

Putting \( \tilde{X} = BX, \tilde{Y} = BY \) and \( \tilde{Z} = BZ \), we have

\[
\tilde{R}(BX, BY) BZ = \tilde{\nabla}_{BX} \tilde{\nabla}_{BY} BZ - \tilde{\nabla}_{BY} \tilde{\nabla}_{BX} BZ - \tilde{\nabla}_{[BX, BY]} BZ.
\]

By virtue of (2.6), (2.8), and (4.2), we get

\[
(4.3) \quad \tilde{R}(BX, BY) BZ = B[R(X, Y) Z + m(X, Z)MY - m(Y, Z)MX]
\]

\[
+ [\nabla_X m](Y, Z) - (\nabla_Y m)(X, Z) + a_\alpha \{b(X) - b(Y)\nabla_m \psi X - \eta(X)\psi Y, Z)] N,
\]

where \( R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \) is the curvature tensor of the quarter-symmetric non-metric connection \( \tilde{\nabla} \).

Substituting

\[
\tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{U}) = g(\tilde{R}(\tilde{X}, \tilde{Y}) \tilde{Z}, \tilde{U})
\]

and

\[
R(X, Y, Z, U) = g(R(X, Y) Z, U).
\]
Then from (4.3), we can easily obtain that

\begin{align}
\tilde{R}(BX, BY, BZ, BU) &= R(X, Y, Z, U) + m(X, Z)h(Y, U) \\
&\quad - m(Y, Z)h(X, U) + a_\alpha(m(Y, Z)g(\psi X, U) - m(X, Z)g(\psi Y, U))
\end{align}

and

\begin{align}
\tilde{R}(BX, BY, BZ, N) &= (\nabla_X m)(Y, Z) - (\nabla_Y m)(X, Z) \\
&\quad + a_\alpha(b(X) - b(Y)) + m(\eta^\alpha(Y)\psi X - \eta^\alpha(X)\psi Y, Z).
\end{align}

Equations (4.4) and (4.5) are the equations of the Gauss and Codazzi with respect to the quarter-symmetric non-metric connection.

5. Submanifolds of co-dimensions 2

Let $M^{n+1}$ be an $(n + 1)$-dimensional differentiable manifold of differentiability class $C^\infty$ and let $M^{n-1}$ be an $(n - 1)$-dimensional manifold immersed in $M^{n+1}$ by the immersion $\tau : M^{n-1} \to M^{n+1}$. We denote the differentiability $d\tau$ of the immersion $\tau$ by $B$, so that the vector field $X$ in the tangent space of $M^{n-1}$ corresponds to a vector field $BX$ in that of $M^{n+1}$. Suppose that $M^{n+1}$ is an almost $r$-paracontact Riemannian manifold with metric tensor $e g$. Then the submanifold $M^{n-1}$ is also an almost $r$-paracontact Riemannian manifold with metric tensor $g$ such that

$$g(\psi X, Y) = \tilde{g}(B\psi X, BY)$$

for arbitrary vector fields $X, Y$ in $M^{n-1}$ [3].

Let the manifolds $M^{n+1}$ and $M^{n-1}$ are both orientable such that

\begin{align}
\tilde{\psi}BX &= B\psi X + a(X)N_1 + b(X)N_2 \\
\tilde{g}(B\psi X, N_1) &= \tilde{g}(B\psi X, N_2) = \tilde{g}(N_1, N_2) = 0 \\
\text{and} \quad \tilde{g}(N_1, N_1) &= \tilde{g}(N_2, N_2) = 1
\end{align}

for arbitrary vector field $X$ in $M^{n-1}$ [6].

We suppose that the enveloping manifold $M^{n+1}$ admits a quarter-symmetric non-metric connection given by [1]

$$\tilde{\nabla}_X \tilde{Y} = \tilde{\nabla}_X \tilde{Y} + \tilde{\eta}^\alpha(\tilde{Y})\tilde{\psi} \tilde{X}$$

for arbitrary vector field $\tilde{X}, \tilde{Y}$ in $M^{n-1}$, $\tilde{\nabla}$ denotes the Levi-Civita connection with respect to the Riemannian metric $\tilde{g}$, $\tilde{\eta}^\alpha$ is a 1-form. Let us now put

\begin{align}
\tilde{\psi}BX &= B\psi X + a(X)N_1 + b(X)N_2
\end{align}
Submanifolds of an almost $r$-paracontact Riemannian manifold

$$\xi_\alpha = B\xi_\alpha + a_\alpha N_1 + b_\alpha N_2,$$

where $a(X)$ and $b(X)$ are 1-forms on $M^{n-1}$, $\xi_\alpha$ is a vector field in the tangent space on $M^{n-1}$ and $a_\alpha$, $b_\alpha$ are functions on $M^{n-1}$. Defined by $\eta^\alpha(N_1) = a_\alpha$, $\eta^\alpha(N_2) = b_\alpha$.

Then we have the following.

**Theorem 5.1.** The connection induced on the submanifold $M^{n-1}$ of co-dimension two of an almost $r$-paracontact Riemannian manifold $M^{n+1}$ with a quarter-symmetric non-metric connection $\nabla$ is also a quarter-symmetric non-metric connection.

**Proof.** Let $\tilde{\nabla}$ be the connection induced on the submanifold $M^{n-1}$ from the connection $\nabla$ on the enveloping manifold with respect to unit normal vectors $N_1$ and $N_2$, then we have \([9]\)

$$\tilde{\nabla}_B X Y = B(\nabla_X Y) + h(X, Y)N_1 + k(X, Y)N_2$$

for arbitrary vector fields $X$ and $Y$ in $M^{n-1}$, where $h$ and $k$ are the second fundamental tensors of $M^{n-1}$. Similarly, if $\nabla$ be the connection induced on $M^{n-1}$ from the quarter-symmetric non-metric connection $\nabla$ on $M^{n+1}$ we have

$$\nabla_B X Y = B(\nabla_X Y) + m(X, Y)N_1 + n(X, Y)N_2,$$

where $m$ and $n$ being tensor fields of type $(0,2)$ of the submanifold $M^{n-1}$.

In view of equation (2.1), we have

$$\tilde{\nabla}_B X Y = \tilde{\nabla}_B X Y + \tilde{\eta}^\alpha(BY)\tilde{\psi}(BX).$$

Using (5.1), (5.2) and (5.3), we have

$$B(\nabla_X Y) + m(X, Y)N_1 + n(X, Y)N_2 = B(\nabla_X Y) + h(X, Y)N_1 + k(X, Y)N_2 + \eta^\alpha(Y)(B\psi X + a(X))N_1 + b(X)N_2,$$

where $\tilde{\eta}^\alpha(BY) = \eta^\alpha(Y)$ and $\tilde{\psi}(BX) = B\psi X + a(X)N_1 + b(X)N_2$.

Comparing tangential and normal parts we get

$$\nabla_X Y = \tilde{\nabla}_X Y + \eta^\alpha(Y)\psi X,$$

$$m(X, Y) = h(X, Y) + a(X)\eta^\alpha(Y),$$

$$n(X, Y) = k(X, Y) + b(X)\eta^\alpha(Y).$$

Thus we have

$$\nabla_X Y - \nabla_Y X - [X, Y] = \eta^\alpha(Y)\psi X - \eta^\alpha(X)\psi Y.$$
Hence the connection $\nabla$ induced on $M^{n-1}$ is quarter-symmetric non-metric connection.

6. Totally geodesic and totally umbilical submanifolds

Let $X_1, X_2, ..., X_{n-1}$ be $(n-1)$-orthonormal vector fields on the submanifold $M^{n-1}$. Then the function
\[
\frac{1}{2(n-1)} \sum_{i=1}^{n-1} [h(X_i, X_i) + k(X_i, X_i)]
\]
is called the \textit{mean curvature} of $M^{n-1}$ with respect to the Riemannian connection $\hat{\nabla}$ and
\[
\frac{1}{2(n-1)} \sum_{i=1}^{n-1} [m(X_i, X_i) + n(X_i, X_i)]
\]
is called the \textit{mean curvature} of $M^{n-1}$ with respect to the quarter-symmetric non-metric connection $\nabla$ [6].

From this we have the following definitions.

\textbf{Definition 6.1.} If $h$ and $k$ vanish separately, the submanifold $M^{n-1}$ is called \textit{totally geodesic} with respect to the Riemannian connection $\hat{\nabla}$.

\textbf{Definition 6.2.} The submanifold $M^{n-1}$ is called \textit{totally umbilical} with respect to the connection $\nabla$ if $h$ and $k$ are proportional to the metric tensor $g$.

We call $M^{n-1}$ is \textit{totally geodesic} and \textit{totally umbilical} with respect to the quarter-symmetric non-metric connection $\nabla$ according as the function $m$ and $n$ vanish separately and are proportional to the metric tensor $g$ respectively.

\textbf{Theorem 6.3.} The mean curvature of $M^{n-1}$ with respect to the Riemannian connection $\hat{\nabla}$ coincides with that of $M^{n-1}$ with respect to the quarter-symmetric non-metric connection $\nabla$ if and only if
\[
\sum_{i=1}^{n-1} [\eta^\alpha(Y_i)(a(X_i) + b(X_i))] = 0.
\]

\textit{Proof.} In view of (5.4), we have
\[
m(X_i, X_i) + n(X_i, X_i) = h(X_i, X_i) + k(X_i, X_i) + \eta^\alpha(Y_i)(a(X_i) + b(X_i)).
\]
Summing up for $i = 1, 2, \ldots, (n - 1)$ and then divide it by $2(n - 1)$, we get
\[ \frac{1}{2(n - 1)} \sum_{i=1}^{n-1} [m(X_i, X_i) + n(X_i, X_i)] = \frac{1}{2(n - 1)} \sum_{i=1}^{n-1} [h(X_i, X_i) + k(X_i, X_i)] \]
if and only if \[ \sum_{i=1}^{n-1} [\eta^a(Y_i)(a(X_i) + b(X_i))] = 0, \]
which proves our assertion.

**Theorem 6.4.** The submanifold $M^{n-1}$ is totally geodesic with respect to the Riemannian connection $\nabla$ if and only if it is totally geodesic with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$ provided that $a(X) = 0$ and $b(X) = 0$.

**Proof.** The proof follows easily from equations (5.4)\(_a\) and (5.4)\(_b\).

### 7. Curvature tensor and Weingarten equations

For the Riemannian connection $\nabla$, the Weingarten equations are given by [11]

\[ (7.1)\(_a\) \quad \tilde{\nabla}_{BX}N_1 = -BHX + l(X)N_2, \]
\[ (7.1)\(_b\) \quad \tilde{\nabla}_{BX}N_2 = -BKX - l(X)N_1, \]

where $H$ and $K$ are tensor fields of type (1,1) such that $g(HX,Y) = h(X,Y)$ and $g(KX,Y) = k(X,Y)$. Also making use of (2.1), (2.2) and (7.1)\(_a\), we get
\[ \tilde{\nabla}_{BX}N_1 = -B(H - a_\alpha \psi)X + a_\alpha(a(X)N_1 + (b(X)N_2) + l(X)N_2, \]
\[ (7.2) \quad \tilde{\nabla}_{BX}N_1 = -B M_X + a_\alpha(a(X)N_1 + (b(X)N_2) + l(X)N_2, \]

where
\[ M_X =HX - a_\alpha \psi X. \]

Similarly, from (2.1), (2.2) and (7.1)\(_b\), we get
\[ (7.3) \quad \tilde{\nabla}_{BX}N_2 = -B M_X + a_\alpha(a(X)N_1 + (b(X)N_2) - l(X)N_1, \]

where
\[ M_X = KX - b_\alpha \psi X. \]
Equations (7.2) and (7.3) are the Weingarten equations with respect to the quarter-symmetric non-metric connection $\nabla$.

8. Riemannian curvature tensor for quarter-symmetric non-metric connection.

Let $\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}$ be the Riemannian curvature tensor of the enveloping manifold $M^{n+1}$ with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$, then

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = \tilde{\nabla}_{\tilde{X}} \tilde{\nabla}_{\tilde{Y}} \tilde{Z} - \tilde{\nabla}_{\tilde{Y}} \tilde{\nabla}_{\tilde{X}} \tilde{Z} - \tilde{\nabla}_{[\tilde{X}, \tilde{Y}]} \tilde{Z}.$$

Putting $\tilde{X} = BX$, $\tilde{Y} = BY$ and $\tilde{Z} = BZ$, we have

$$\tilde{R}(BX, BY)BZ = \tilde{\nabla}_{BX} \tilde{\nabla}_{BY} BZ - \tilde{\nabla}_{BY} \tilde{\nabla}_{BX} BZ - \tilde{\nabla}_{[BX, BY]} BZ.$$

Using (5.3), we get

$$\tilde{R}(BX, BY)BZ = \tilde{\nabla}_{BX}(B(\nabla_Y Z) + m(Y, Z)N_1 + n(Y, Z)N_2)$$

$$- \tilde{\nabla}_{BY}(B(\nabla_X Z) + m(X, Z)N_1 + n(X, Z)N_2)$$

$$- (B(\nabla_{[X,Y]} Z) + m([X,Y], Z)N_1 + n([X,Y], Z)N_2).$$

Again using (5.3), (5.5), (7.2) and (7.3), we have

$$\tilde{R}(BX, BY)BZ = BR(X, Y, Z) + B(m(X, Z)M_1 Y - m(Y, Z)M_1 X)$$

$$+ n(X, Z)M_2 Y - n(Y, Z)M_2 X + m(\eta^\alpha(Y)\psi X - \eta^\alpha(X)\psi Y, Z)N_1$$

$$+ n(\eta^\alpha(Y)\psi X - \eta^\alpha(X)\psi Y, Z)N_2 + ((\nabla_X m)(Y, Z) - (\nabla_Y m)(X,Z))N_1$$

$$+ (((\nabla_X n)(Y, Z) - (\nabla_Y n)(X,Z))N_2 + l(X)(m(Y, Z)N_2 - n(Y, Z)N_1)$$

$$- l(Y)(m(X, Z)N_2 - n(X, Z)N_1) + a_\alpha((a(X)N_1 + b(X)N_2)m(Y, Z)$$

$$- (a(Y)N_1 + b(Y)N_2)m(X, Z)) + b_\alpha((a(X)N_1 + b(X)N_2)n(Y, Z)$$

$$- (a(Y)N_1 + b(Y)N_2)n(X, Z)),$$

where $R(X, Y, Z)$ is the Riemannian curvature tensor of the submanifold with respect to the quarter-symmetric non-metric connection $\nabla$. 
References


Department of Mathematics
Integral University
Kursi-Road, Lucknow, 226026, India
E-mail: mobinahmad@rediffmail.com

Department of Mathematics
Kookmin University
Seoul 136-702, Republic of Korea
E-mail: jbjun@kookmin.ac.kr

Department of Mathematics
Integral University
Kursi-Road, Lucknow, 226026, India