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MIGHTY FILTERS IN BE-ALGEBRAS

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Abstract. The notion of a mighty (vague) filter in a BE-algebra is introduced, and the relation between a (vague) filter and a mighty (vague) filter are given. We investigate an equivalent condition for a (vague) filter to be mighty, and state an extension property for mighty filter. Also we define the notion of an n-fold mighty filter which is an extended notion of a mighty filter in a BE-algebra. Characterizations of an n-fold mighty filter are given. Extension property for an n-fold mighty filter are provided.

1. Introduction

Several authors from time to time have made a number of generalizations of Zadeh's fuzzy set theory [12]. Of course, the notion of vague set theory introduced by Gau and Buehrer [4] is of interest to us. Using the vague set in the sense of Gau and Buehrer, Biswas [3] studied vague groups. Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [5,6]. In [8]. H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra as a generalization of a BCK-algebra. Jun and Park [7,11] studied vague ideals and vague deductive systems in subtraction algebras. K. J. Lee, Y. H. Kim and Y. U. Cho [9] introduced the notion of vague BCK/BCI-algebras and vague ideals, and investigated their properties. Sun Shin Ahn and Jung Mi Ko [2] introduced the notion of a vague filter in BE-algebra, and investigate some properties of it.

In this paper, we introduce the notions of a mighty filter and a mighty vague filter in a BE-algebra and study the relation between a (vague) filter and a mighty (vague) filter. We provide an equivalent condition for a (vague) filter to be (vague) mighty, and state an extension property for mighty filter. Also we define the notion of an n-fold mighty

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filter which is an extended notion of a mighty filter in a BE-algebra. We investigate characterizations of an n-fold mighty filter and study an extension property for an n-fold mighty filter.

2. Preliminaries

We recall some definitions and results discussed in [8].

An algebra $(X;*,1)$ of type (2, 0) is called a *BE-algebra* if

(BE1) $x*x=1$ for all $x\in X$;
(BE2) $x*1=1$ for all $x\in X$;
(BE3) $1*x=x$ for all $x\in X$;
(BE4) $x*(y*z)=y*(x*z)$ for all $x,y,z\in X$ (*exchange*)

We introduce a relation "$\leq$" on a BE-algebra X by $x\leq y$ if and only if $x*y=1$. A non-empty subset S of a BE-algebra X is said to be a *subalgebra* of X if it is closed under the operation "$*$". Noticing that $x*x=1$ for all $x\in X$, it is clear that $1\in S$. A BE-algebra $(X;*,1)$ is said to be *self distributive* if $x*(y*z)=(x*y)*(x*z)$ for all $x,y,z\in X$.

Definition 2.1.([8]) Let $(X;*,1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is called a *filter* of X if

(F1) $1\in F$;
(F2) $x*y\in F$ and $x\in F$ imply $y\in F$.

Proposition 2.2.([8]) *$(X;*,1)$ is a BE-algebra, then $x*(y*x)=1$ for any $x,y\in X$.*

Example 2.3.([8]) Let $X:=\{1,a,b,c,d,0\}$ be a BE-algebra with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $F_1:=\{1,a,b\}$ is a filter of X, but $F_2:=\{1,a\}$ is not a filter of X, since $a*b\in F_2$ and $a\in F_2$, but $b\notin F_2$.

Proposition 2.4. *Let $(X;*,1)$ be a BE-algebra and let F be a filter of X. If $x \leq y$ and $x \in F$ for any $y \in X$, then $y \in F$.*

Proposition 2.5. *Let $(X;*,1)$ be a self distributive BE-algebra. Then following hold: for any $x, y, z \in X$,*

(i) *if $x \leq y$, then $z * x \leq z * y$ and $y * z \leq x * z$.*
(ii) $y * z \leq (z * x) * (y * z)$.
(iii) $y * z \leq (x * y) * (x * z)$.

A BE-algebra $(X;*,1)$ is said to be *transitive* if it satisfies Proposition 2.5(iii). If a BE-algebra X is transitive, then $y \leq z$ imply $x * y \leq x * z$ and $z * x \leq y * x$ for all $x, y \in X$ ([10]).

Definition 2.6.([3]) A *vague set* A in the universe of discourse U is characterized by two membership functions given by:

(1) A truth membership function

$$t_A : U \to [0,1],$$

and

(2) A false membership function

$$f_A : U \to [0,1],$$

where $t_A(u)$ is a lower bound of the grade of membership of u derived from the "evidence for u", and $f_A(u)$ is a lower bound on the negation of u derived from the "evidence against u", and

$$t_A(u) + f_A(u) \leq 1.$$

Thus the grade of membership of u in the vague set A is bounded by a subinterval $[t_A(u), 1 - f_A(u)]$ of $[0,1]$. This indicates that if the actual grade of membership is $\mu(u)$, then

$$t_A(u) \leq \mu(u) \leq 1 - f_A(u).$$

The vague set A is written as

$$A = \{\langle u, [t_A(u), f_A(u)]\rangle | u \in U\},$$

where the interval $[t_A(u), 1 - f_A(u)]$ is called the *vague value* of u in A and is denoted by $V_A(u)$.

3. Mighty filters

In what follows, let X be a BE-algebra unless otherwise specified.

Proposition 3.1. Let F be a filter of a BE-algebra X. Then F is a subalgebra of X.

Proof. By Proposition 2.2, we have $y * (x * y) = 1$ for any $x, y \in X$. Since F is a filter of X, we have $x * y \in F$ for any $x, y \in F$. This completes the proof. □

Proposition 3.2. *Let F be a non-empty subset of a BE-algebra X. Then F is a filter of X if and only if for all $x, y \in F$ and $z \in X$, $x \leq y * z$ implies $z \in F$.*

Proof. Let F be a filter and let $x, y \in F$ and $z \in X$. If $x \leq y * z$, then $x * (y * z) = 1 \in F$. Since $x \in F$ and F is a filter, we have $y * z \in F$. Using (F2), we obtain $z \in F$.

Conversely, we assume that for all $x, y \in F$ and $z \in X$, $x \leq y * z$ implies $z \in F$. Let $a \in F$. Then $a * (a * 1) = a * 1 = 1$. By assumption, we have $1 \in F$. If $x \in F$ and $x*z \in F$, then we obtain $(x*z)*(x*z) = 1$. Hence $z \in F$. This F is a filter of X. □

Definition 3.3. A non-empty subset F of a BE-algebra X is called a *mighty filter* of X if it satisfies (F1) and

(F3) $z*(y*x) \in F$ and $z \in F$ imply $((x*y)*y)*x \in F$ for all $x, y, z \in X$.

Example 3.4. Let $X := \{1, a, b, c, d, 0\}$ be a set with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	b	c	b	c
b	1	a	1	b	a	d
c	1	a	1	1	a	a
d	1	1	1	b	1	b
0	1	1	1	1	1	1

It is easy to check that X is a (transitive) BE-algebra and $F := \{1, b, c\}$ is a mighty filter of X.

Theorem 3.5. *Every mighty filter of a BE-algebra X is a filter of X.*

Proof. Let F be a mighty filter of X and let $z*x \in F$ and $z \in F$. Then $z*(1*x) = z*x \in F$. It follows from (F3) that $x = ((x*1)*1)*x \in F$. Hence F is a filter. □

The converse of Theorem 3.5 is not true in general as seen the following example.

Example 3.6. Let $X := \{1, a, b, c, d\}$ be a set with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then X is a self distributive BE-algebra ([8]) and $F := \{1\}$ is a filter of X, but not a mighty filter of X since $1*(c*a) = 1 \in F$ and $((a*c)*c)*a = a \notin F$.

We give an equivalent condition for a filter to be a mighty filter.

Theorem 3.7. *A filter F is mighty if and only if it satisfies:*

($*$) $y*x \in F$ *implies* $((x*y)*y)*x \in F$ *for all* $x, y \in X$.

Proof. Assume that F is a mighty filter of X and let $x, y \in X$ be such that $y*x \in F$. Then $1*(y*x) = y*x \in F$ and $1 \in F$. It follows from (F3) that $((x*y)*y)*x \in F$ for all $x, y \in X$.

Conversely, let F be a filter of X satisfying ($*$) and let $x, y, z \in X$ be such that $z*(y*x) \in F$ and $z \in F$. Then $y*x \in F$ by (F2) and hence $((x*y)*y)*x \in F$ by ($*$). Thus F is a mighty filter of X. □

Theorem 3.8. [Extension property for a mighty filter] *Let F and G be filters of a transitive BE-algebra X such that $F \subseteq G$. If F is mighty, then so is G.*

Proof. Let $x, y \in X$ be such that $y*x \in G$. Then $y*((y*x)*x) = (y*x)*(y*x) = 1 \in F$. Since F is mighty, it follows from Theorem 3.7 that $((((y*x)*x)*y)*y)*((y*x)*x) \in F$ so that $(y*x)*(((((y*x)*x)*y)*y)*x) \in G$. Since G is a filter and $y*x \in G$, we have $((((y*x)*x)*y)*y)*x \in G$. Since X is transitive, we get

$$\begin{aligned}[((((y*x)*x)*y)*y)*x]*[((x*y)*y)*x] \\ &\geq ((x*y)*y)*((((y*x)*x)*y)*y) \\ &\geq (((y*x)*x)*y)*(x*y) \\ &\geq x*((y*x)*x) \\ &= (y*x)*(x*x) \\ &= (y*x)*1 = 1.\end{aligned}$$

It follows from Proposition 3.2 that $((x*y)*y)*x \in G$. Hence, by Theorem 3.7, G is a mighty filter of X. □

Corollary 3.9. *Every filter of a transitive BE-algebra X is mighty if and only if the filter $\{1\}$ is mighty.*

Proof. Straightforward. □

Let F be a filter of a transitive BE-algebra X. Define a relation ρ on X by $(x,y) \in \rho$ if and only if $x*y \in F$ and $y*x \in F$. Then ρ is a congruence relation on X (See [10]). Denote $X/\rho := \{[x]_\rho | x \in X\}$, where $[x]_\rho := \{y \in X | (x,y) \in \rho\}$. We define a binary operation $*'$ on X/ρ by $[x]_\rho *' [y]_\rho := [x*y]_\rho$. This definition is well defined since ρ is a congruence relation. Also $[1]_\rho = F$.

Proposition 3.10. *$(X/\rho; *', [1]_\rho)$ is a transitive BE-algebra.*

Proof. By Proposition 5.4 of [10], $(X/\rho; *', [1]_\rho)$ is a BE-algebra. It is easy to check that X/ρ is transitive. This completes the proof. □

Theorem 3.11. *A filter F of a transitive BE-algebra X is mighty if and only if every filter of the quotient algebra X/ρ is mighty.*

Proof. Assume that F is a mighty filter of X and let $x,y \in X$ be such that $[x]_\rho *' [y]_\rho = [1]_\rho$. Then $x*y \in F$ and so $((y*x)*x)*y \in F$ by Theorem 3.7. Hence $(([y]_\rho *' [x]_\rho) *' [x]_\rho) *' [y]_\rho = [((y*x)*x)*y]_\rho = [1]_\rho$ which proves that $\{[1]_\rho\}$ is a mighty filter of X/ρ. By Corollary 3.9, every filter of X/ρ is mighty.

Conversely, suppose that every filter of X/ρ is mighty and let $x,y \in X$ be such that $y*x \in F$. Then $[y]_\rho *' [x]_\rho = [y*x]_\rho = [1]_\rho$. Since $\{[1]_\rho\}$ is a mighty filter of X/ρ, it follows from Theorem 3.7 that $[((x*y)*y)*x]_\rho = (([x]_\rho *' [y]_\rho) *' [y]_\rho) *' [x]_\rho = [1]_\rho$, i.e., $((x*y)*y)*x \in F$. Hence F is a mighty filter of X by Theorem 3.7. □

4. Mighty vague filters

For our discussion, we shall use the following notations, which are given in [3], on interval arithmetic.

Let $I[0,1]$ denote the family of all closed subintervals of $[0,1]$. We define the term "imax" to mean the maximum of two intervals as

$$\text{imax}(I_1, I_2) = [\max(a_1, a_2), \max(b_1, b_2)],$$

where $I_1 = [a_1, b_1], I_2 = [a_2, b_2] \in I[0,1]$. Similarly, we define "imin". The concept of "imax" and "imin" could be extended to define "isup" and "iinf" of infinite number of elements of $I[0,1]$. It is obvious that $L = \{I[0,1], \text{isup}, \text{iinf}, \leq\}$ is a lattice with universal bounds $[0,0]$ and $[1,1]$ ([3]).

Definition 4.1.([2]) A vague set A of a BE-algebra X is called a *vague filter* of X if the following conditions are true:

(c1) $(\forall x \in X)\ (V_A(1) \succeq V_A(x))$,
(c2) $(\forall x, y \in X)\ (V_A(y) \succeq \text{imin}\{V_A(x * y), V_A(x)\})$,

that is,

$$t_A(1) \geq t_A(x), 1 - f_A(1) \geq 1 - f_A(x)$$

and

$$t_A(y) \geq \min\{t_A(x * y), t_A(x)\},$$
$$1 - f_A(y) \geq \min\{1 - f_A(x * y), 1 - f_A(x)\}$$

for all $x, y \in X$.

Proposition 4.2.([2]) *Every vague filter A of a BE-algebra X satisfies the following properties:*

(i) $(\forall x, y \in X)(x \leq y \Rightarrow V_A(x) \preceq V_A(y))$,
(ii) $(\forall x, y, z \in X)(V_A(x * z) \succeq \textit{imin}\{V_A(x * (y * z)), V_A(y)\})$.

Theorem 4.3.([2]) *Let A be a vague set of a BE-algebra X. Then A is a vague filter of X if and only if it satisfies*

$$(\forall x, y, z \in X)(z \leq x * y \Rightarrow V_A(y) \succeq \textit{imin}\{V_A(x), V_A(z)\}).$$

Definition 4.4. A vague set A of a BE-algebra X is called a *mighty vague filter* of X if it satisfies (c1) and

(c3) $(\forall x, y, z \in X)\ (V_A(((x * y) * y) * x) \succeq \text{imin}\{V_A(z * (y * x)), V_A(z)\})$, that is,

$$t_A(1) \geq t_A(x), 1 - f_A(1) \geq 1 - f_A(x)$$

and

$$t_A(((x * y) * y) * x) \geq \min\{t_A(z * (y * x)), t_A(z)\},$$
$$1 - f_A(((x * y) * y) * x) \geq \min\{1 - f_A(z * (y * x)), 1 - f_A(z)\}$$

for all $x, y, z \in X$.

Let us illustrate this definition using the following example.

Example 4.5. Consider a BE-algebra $X = \{1, a, b, c, d, 0\}$ as in Example 3.4. Let A be a vague set in X defined as follows:

$$A := \{\langle 1, [0.8, 0.2]\rangle, \langle a, [0.5, 0.3]\rangle, \langle b, [0.8, 0.2]\rangle, \langle c, [0.8, 0.2]\rangle, \langle d, [0.5, 0.3]\rangle, \langle 0, [0.5, 0.3]\rangle\}.$$

It is routine to verify that A is a mighty vague filter of X.

Theorem 4.6. *Every mighty vague filter is a vague filter.*

Proof. Let A be a mighty vague filter of a BE-algebra X. If we take $y := 1$ in (c3), then we obtain (c2). Hence A is a vague filter of X. □

The converse of Theorem 4.6 is not true in general as the following example.

Example 4.7. Consider a BE-algebra $X = \{1, a, b, c, d\}$ as in Example 3.6. Let B be a vague set in X defined as follows:

$$B := \{\langle 1, [0.7, 0.2]\rangle, \langle a, [0.5, 0.3]\rangle, \langle b, [0.5, 0.3]\rangle, \langle c, [0.5, 0.3]\rangle, \langle d, [0.5, 0.3]\rangle\}.$$

It is routine to verify that B is a vague filter of X. But it is not a mighty vague filter of X, since

$$V_B(((a * c) * c) * a) = V_B(a) \not\succeq V_B(1) = \mathrm{imin}\{V_B(1 * (c * a)), V_B(1)\}).$$

Now, we give an equivalent condition for every vague filter to be a mighty vague filter.

Theorem 4.8. *For any vague filter A of a BE-algebra X, the following are equivalent:*

(i) *A is a mighty vague filter of X.*
(ii) $(\forall x, y \in X)(V_A(((x * y) * y) * x) \succeq V_A(y * x)$.

Proof. (i)⇒(ii) Assume that A is a mighty vague filter of X. Putting $z := 1$ in (c3) and , we have

$$\begin{aligned} V_A(((x*y)*y)*x) &\succeq \text{imin}\{V_A(1*(y*x)), V_A(1)\} \\ &= \text{imin}\{V_A(y*x)), V_A(1)\} \\ &= V_A(y*x)). \end{aligned}$$

Hence (ii) holds.
(ii)⇒(i) Suppose that (ii) holds. Using (c2), we have

$$\begin{aligned} V_A(((x*y)*y)*x) &\succeq V_A(y*x) \\ &\succeq \text{imin}\{V_A(z*(y*x)), V_A(z)\} \end{aligned}$$

for all $x, y, z \in X$. Thus A is a mighty vague filter of X. □

5. n-fold mighty filters

In what follows, let n and X denote a positive integer and a BE-algebra, respectively, unless otherwise specified. For any elements x and y of X, let $x^n * y$ denote $x * (\cdots * (x * (x * y)) \cdots)$ in which x occurs n times, and $x^0 * y = y$.

Definition 5.1. A non-empty subset F of a BE-algebra X is called an *n-fold mighty filter* of X if it satisfies (F1) and

(F4) $z * (y * x) \in F$ and $z \in F$ imply $((x^n * y) * y) * x \in F$ for all $x, y, z \in X$.

Definition 5.2. A non-empty subset F of a BE-algebra X is said to be a *weak n-fold mighty filter* of X if it satisfies (F1) and

(F5) $z * ((y^n * x) * x) \in F$ and $z \in F$ imply $(x * y) * y \in F$ for all $x, y, z \in X$.

Putting $y = 1$ and $y = x$ in (F4) and (F5), respectively, and using (BE1), (BE2) and (BE3), we know that every (weak) n-fold mighty filter is a filter.

Example 5.3. Consider a BE-algebra $X = \{1, a, b, c, d, 0\}$ which is given in Example 2.3. It is easy to check that $F := \{1, a\}$ is a 2-fold mighty filter of X.

Theorem 5.4. *Let F be a filter of a BE-algebra X. Then*

(i) *F is an n-fold mighty filter of X if and only if $((x^n * y) * y) * x \in F$ for all $x, y \in X$ with $y * x \in F$.*

(ii) *F is a weak n-fold mighty filter of X if and only if $(x*y)*y \in F$ for all $x, y \in X$ with $(y^n * x)*x \in F$.*

Proof. (i) Assume that F is an n-fold mighty filter of X and let $x, y \in X$ be such that $y*x \in F$. Then $1*(y*x) = y*x$ and $1 \in F$. Using (F4), we have $((x^n*y)*y)*x \in F$.

Conversely, let F be a filter of X such that $((x^n*y)*y)*x \in F$ for all $x, y \in X$ with $y*x \in F$. Let $x, y, z \in X$ be such that $z*(y*x) \in F$ and $z \in F$. By (F2), we have $y*x \in F$. By assumption, we have $((x^n*y)*y)*x \in F$. Thus F is an n-fold mighty filter of X.
(ii) Let F be a filter of X such that $z*((y^n*x)*x) \in F$ and $z \in F$. By (F2), we have $(y^n*x)*x \in F$. Using assumption, we get $(x*y)*y \in F$. Hence F is a weak n-fold mighty filter of X.

Conversely, assume that F is a weak n-fold mighty filter of X. Let $x, y \in X$ with $(y^n*x)*x \in F$. Then $1*((y^n*x)*x) = (y^n*x)*x \in F$ and $1 \in F$. By (F5), we have $(x*y)*y \in F$. This competes the proof. □

Corollary 5.5. *Let F be a filter of a BE-algebra X. Then F is mighty if and only if $((x*y)*y)*x \in F$ for all $x, y \in X$ with $y*x \in F$.*

Proof. Put in $n = 1$ in Theorem 5.4(i). □

Theorem 5.6. [Extension Property for an n-fold mighty filter] *Let F and G be filters of a transitive BE-algebra X such that $F \subseteq G$. If F is n-fold mighty, then so is G.*

Proof. Let $x, y \in X$ be such that $y*x \in G$. Setting $w = (y*x)*x$, then $y*w = y*((y*x)*x) = (y*x)*(y*x) = 1 \in F$. Since F is n-fold, it follows from Theorem 5.4(i) that

$$\begin{aligned}(y*x)*(((w^n*y)*y)*x) &= ((w^n*y)*y)*((y*x)*x)\\ &= ((w^n*y)*y)*w \in F \subseteq G,\end{aligned}$$

which implies from (F2) that $((w^n*y)*y)*x \in G$. Since $x \le w$, we have $w^n*y \le x^n*y$, and so $((w^n*y)*y)*x \le ((x^n*y)*y)*x$. It follows from Proposition 2.4 that $((x^n*y)*y)*x \in G$. Hence G is an n-fold mighty filter of X by Theorem 5.4(i). □

Corollary 5.7. *Every filter of a transitive BE-algebra X is n-fold mighty filter if and only if the filter $\{1\}$ is n-fold mighty.*

Proof. Straightforward. □

Corollary 5.8. *Let F and G be filters of a transitive BE-algebra X such that $F \subseteq G$. If F is mighty, then so is G.*

Proof. Put $n = 1$ in Theorem 5.6. □

Let F be a filter of a transitive BE-algebra. Define a relation ρ on X by $(x, y) \in \rho$ if and only if $x * y \in F$ and $y * x \in F$. Then ρ is a congruence relation on X (see [10]). Denote $X/\rho := \{[x]_\rho | x \in X\}$, where $[x]_\rho := \{y \in X | (x, y) \in \rho\}$. Then $(X/\rho; *', [1]_\rho)$ is a transitive B-algebra (see Proposition 3.10), where $[x]_\rho *' [y]_\rho := [x * y]_\rho$.

Theorem 5.9. *A filter F of a transitive BE-algebra X is n-fold mighty if and only if every filter of the quotient algebra X/ρ is n-fold mighty.*

Proof. Assume that F is an n-fold mighty filter of a transitive BE-algebra X and let $x, y \in X$ be such that $[x]_\rho *' [y]_\rho = [1]_\rho$. Then $x * y \in F$ and so $((y^n * x) * x) * y \in F$ by Theorem 5.4(i). Hence $(([y]_\rho^n *' [x]_\rho) *' [x]_\rho) *' [y]_\rho = [((y^n * x) * x) * y]_\rho = [1]_\rho$ which proves that $\{[1]_\rho\}$ is an n-fold mighty filter of X/ρ. By Corollary 5.7, every filter of X/ρ is n-fold mighty.

Conversely, suppose that every filter of X/ρ is n-fold mighty and let $x, y \in X$ be such that $y * x \in F$. Then $[y]_\rho *' [x]_\rho = [y * x]_\rho = [1]_\rho$. Since $\{[1]_\rho\}$ is an n-fold mighty filter of X/ρ, it follows from Theorem 5.4(i) that $[((x^n * y) * y) * x]_\rho = ([x]_\rho^n *' [y]_\rho) *' [y]_\rho) *' [x]_\rho = [1]_\rho$, i.e., $((x^n * y) * y) * x \in F$. Hence F is an n-fold mighty filter of X by Theorem 5.4(i). □

Corollary 5.10. *A filter F of a transitive BE-algebra X is mighty if and only if every filter of the quotient algebra X/ρ is mighty.*

Proof. Put $n = 1$ in Theorem 5.9. □

A BE-algebra X is said to be *n-fold mighty* if it satisfies the equality $((x^n * y) * y) * x = y * x$ for all $x, y \in X$. Note that, in an n-fold mighty BE-algebra, the notion of filters, n-fold mighty filters, and weak n-fold mighty filters is coincides.

Proposition 5.11. *If X is an n-fold mighty BE-algebra, then $(x^n * y) * y \leq (y * x) * x$ for all $x, y \in X$.*

Proof. Let X be an n-fold mighty BE-algebra. Then

$$((x^n * y) * y) * ((y * x) * x) = (y * x) * (((x^n * y) * y) * x) = (y * x) * (y * x) = 1$$

for all $x, y \in X$. Hence $(x^n * y) * y \leq (y * x) * x$ for all $x, y \in X$. □

Proposition 5.12. *Let X be a transitive BE-algebra. Then the following are equivalent.*

(i) $(x^n * y) * y \leq (y * x) * x, \forall x, y \in X$.

(ii) $x^n * z \leq y * z, z \leq x \Rightarrow y \leq x$.

(iii) $x^n * z \leq y * z, z \leq x, y \Rightarrow y \leq x$.
(iv) $y \leq x \Rightarrow (x^n * y) * y \leq x$.

Proof. (i)⇒(ii) Let $x, y, z \in X$ be such that $x^n * z \leq y * z$ and $z \leq x$. It follows from (i) that

$$1 = (x^n * z) * (y * z) = y * ((x^n * z) * z) \leq y * ((z * x) * x) = y * (1 * x) = y * x.$$

Hence $y * x = 1$, i.e., $y \leq x$.
(ii)⇒(iii) Trivial.
(iii)⇒(iv) Let $x, y \in X$ be such that $y \leq x$. Note that $y \leq (x^n * y) * y$ and $x^n * y \leq ((x^n * y) * y) * y$. It follows from (iii) that $(x^n * y) * y \leq x$.
(iv)⇒(i) Since $x \leq (y * x) * x$, we have $((y * x) * x)^n * y \leq x^n * y$ by the mathematical induction. Since $y \leq (y * x) * x$, it follows from (iv) that $(x^n * y) * y \leq (((y * x) * x)^n * y) * y \leq (y * x) * x$. This completes the proof. □

Proposition 5.13. *If a BE-algebra X is n-fold mighty, then its trivial filter $\{1\}$ is n-fold mighty.*

Proof. Straightforward. □

Theorem 5.14. *Every n-fold mighty filter of a transitive BE-algebra X is a weak n-fold mighty filter of X.*

Proof. Let F be an n-fold mighty filter of a transitive BE-algebra X. Then X/ρ is n-fold mighty. Let $x, y \in X$ be such that $(y^n * x) * x \in F$. Using Proposition 5.11, we have

$$\begin{aligned} [(x * y) * y]_\rho =& ([x]_\rho *' [y]_\rho) *' [y]_\rho \\ \geq& ([y]_\rho^n *' [x]_\rho) *' [x]_\rho \\ =& [(y^n * x) * x]_\rho = [1]_\rho, \end{aligned}$$

and so $[(x * y) * y]_\rho = [1]_\rho$, i.e, $(x * y) * y \in F$. It follows from Theorem 5.4(ii) that F is a weak n-fold mighty filter of X. □

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