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PERIODICITIES OF SOME HYBRID CELLULAR AUTOMATA WITH PERIODIC BOUNDARY CONDITION

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Abstract. We investigate periodicities of some hybrid cellular automata configured with rule 60 and 102 and periodic boundary condition.

1. Introduction

Cellular automata have been demonstrated by many researchers to be a good computational model for physical systems simulation since the concept of cellular automata first introduced by John Von Neumann [7] in the 1950's. Cellular automata with null boundary condition have been studied by many researchers. Some researches about cellular automata with periodic boundary condition mainly focused on reversibility [1, 2, 8]. And characterizations of powers of characteristic matrices of some uniform cellular automata with periodic boundary condition have been studied [3,4].

In this note, we will investigate periodicities of some hybrid cellular automata configured with rule 60 and 102 and periodic boundary condition.

2. Terminologies

In this section, we will introduce some terminologies shall be used in this note.

A cellular automaton (CA) is an array of sites (cells) where each site is in any one of the permissible states. At each discrete time step (clock cycle) the evolution of a site value depends on some rule (the

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combinational logic) which is a function of the present states of its k neighbors for a k-neighborhood CA. For a 2-state 3-neighborhood CA, the evolution of the (i)th cell can be represented as a function of the present states of $(i-1)$th, (i)th, and $(i+1)$th cells as: $x_i(t+1) = f\{x_{i-1}(t), x_i(t), x_{i+1}(t)\}$, where f represents the combinational logic. For such a CA, the modulo-2 logic is always applied.

For a 2-state 3-neighborhood CA there are 2^3 distinct neighborhood configurations and 2^{2^3} distinct mappings from all these neighborhood configurations to the next states, each mapping representing a CA rule. The CA, characterized by a rule known as rule 60, specifies an evolution from the neighborhood configurations to the next states as;

111	110	101	100	011	010	001	000
0	0	1	1	1	1	0	0

.

The rule name 60 comes from that 00111100 in binary system is 60 in decimal system. The corresponding combinational logic of rule 60 is

$$x_i(t+1) = x_{i-1}(t) \oplus x_i(t),$$

that is, the next state of (i)th cell depends on the present states of its left and self neighbors.

And the CA, characterized by a rule known as rule 102, specifies an evolution from the neighborhood configurations to the next states as;

111	110	101	100	011	010	001	000
0	1	1	0	0	1	1	0

.

The rule name 102 comes from that 01100110 in binary system is 102 in decimal system. The corresponding combinational logic of rule 102 is

$$x_i(t+1) = x_i(t) \oplus x_{i+1}(t),$$

that is, the next state of (i)th cell depends on the present states of self and its right neighbors.

If in a CA the same rule applies to all cells, then the CA is called a uniform CA; otherwise the CA is called a hybrid CA. There can be various boundary conditions; namely, null (where extreme cells are connected to logic ‘0’), intermediate (where the second right cell of the leftmost cell of a 3-neighborhood CA is assumed to be the left neighbor of the leftmost cell of the CA and the second left cell of the rightmost cell of the CA is assumed to be the right neighbor of the rightmost cell of the CA), periodic (where extreme cells are adjacent), etc. If the rule of a CA cell involves only XOR logic, then the rule is called a linear rule. A CA with all the cells having linear rules is called a linear CA. And the number of cells of a CA is called the length of a CA.

The characteristic matrix T of a CA is the transition matrix of the CA. The next state $f_{t+1}(x)$ of a linear CA is given by $f_{t+1}(x) = T \times f_t(x)$, where $f_t(x)$ is the current state and t is the time step.

3. Periodicities of cellular automata

We will start this section with periodicities of the most simple type of hybrid CA's with rules 102 and 60 and periodic boundary condition.

PROPOSITION 3.1. *Let H be a hybrid linear CA of length ℓ configured with rules* 60 *and* 102 *and periodic boundary condition. Assume that the rule applied to the first m cells of H is* 102 *and the rule applied to the second n cells of H is* 60 *with $m \geq 1$, $n \geq 1$ and $\ell = m + n$. If a is a non-negative integer so that $max\{m, n\} \leq 2^a + 1$, then $T^{2+2^a} = T^2$ where T is the characteristic matrix of H.*

Proof. The same result with null boundary condition was proved in [6]. Since the rule applied to the first m cells of H is 102 and the rule applied to the second n cells of H is 60 and since there is no other cell of H, the characteristic matrix of H with periodic boundary condition is the same as the characteristic matrix of H with null boundary condition. Thus we have the proposition. □

To get another result from Proposition 3.1, we need to describe explicitly the characteristic matrix T of H in Proposition 3.1;

$$T = \begin{pmatrix}
\cdot & \cdot & & & & & & & & & & \\
& \cdot & \cdot & & & & & & & & & \\
& & \cdot & \cdot & & & & & & & & \\
& & & 1 & 1 & & & & & & & \\
& & & & 1 & 1 & & & & & & \\
& & & & & 1 & 1 & & & & & \\
& & & & & 1 & 1 & & & & & \\
& & & & & & 1 & 1 & & & & \\
& & & & & & & 1 & 1 & & & \\
& & & & & & & & \cdot & \cdot & & \\
& & & & & & & & & \cdot & \cdot & \\
& & & & & & & & & & \cdot & \cdot
\end{pmatrix},$$

where the blank entries are all zero.

PROPOSITION 3.2. *Let K be a hybrid linear CA of length ℓ configured with rules* 60 *and* 102 *and periodic boundary condition. Assume that* *the rule applied to the first m cells of K is* 60 *and the rule applied to the second n cells of K is* 102 *with $m \geq 1$, $n \geq 1$ and $\ell = m + n$. If a is a non-negative integer so that $max\{m, n\} \leq 2^a + 1$, then $U^{2+2^a} = U^2$ where U is the characteristic matrix of K.*

Proof. At first, the characteristic matrix U is given by

$$\begin{pmatrix}
1 & & & & & & & & & & & & 1 \\
1 & 1 & & & & & & & & & & & \\
& \cdot & \cdot & & & & & & & & & & \\
& & \cdot & \cdot & & & & & & & & & \\
& & & \cdot & \cdot & & & & & & & & \\
& & & & 1 & 1 & & & & & & & \\
& & & & & 1 & 1 & & & & & & \\
& & & & & & & 1 & 1 & & & & \\
& & & & & & & & 1 & 1 & & & \\
& & & & & & & & & \cdot & \cdot & & \\
& & & & & & & & & & \cdot & \cdot & \\
& & & & & & & & & & & \cdot & \cdot \\
& & & & & & & & & & & 1 & 1 \\
1 & & & & & & & & & & & & 1
\end{pmatrix},$$

where the blank entries are all zero. So if $f_t(x) = (x_1, \cdots, x_{(m+n)})^T$ is the current state of K, then the next state $f_{(t+1)}(x) = U \times f_{(t)}(x)$ of K is given by

$$\begin{pmatrix}
1 & & & & & & & & & & & & 1 \\
1 & 1 & & & & & & & & & & & \\
& \cdot & \cdot & & & & & & & & & & \\
& & \cdot & \cdot & & & & & & & & & \\
& & & \cdot & \cdot & & & & & & & & \\
& & & & 1 & 1 & & & & & & & \\
& & & & & 1 & 1 & & & & & & \\
& & & & & & & 1 & 1 & & & & \\
& & & & & & & & 1 & 1 & & & \\
& & & & & & & & & \cdot & \cdot & & \\
& & & & & & & & & & \cdot & \cdot & \\
& & & & & & & & & & & \cdot & \cdot \\
& & & & & & & & & & & 1 & 1 \\
1 & & & & & & & & & & & & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{(m-1)} \\ x_m \\ x_{(m+1)} \\ x_{(m+2)} \\ \cdot \\ \cdot \\ \cdot \\ x_{(m+n-1)} \\ x_{(m+n)}
\end{pmatrix},$$

thus

$$f_{(t+1)}(x) = \begin{pmatrix} x_1 + x_{(m+n)} \\ x_1 + x_2 \\ x_2 + x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_{(m-1)} + x_m \\ x_{(m+1)} + x_{(m+2)} \\ x_{(m+2)} + x_{(m+3)} \\ \cdot \\ \cdot \\ \cdot \\ x_{(m+n-1)} + x_{(m+n)} \\ x_{(m+n)} + x_1 \end{pmatrix}.$$

Now let $g_t(x) = (x_{(m+1)}, \cdots, x_{(m+n)}, x_1, \cdots, x_m)^T$ and

$$g_{(t+1)}(x) = \begin{pmatrix} x_{(m+1)} + x_{(m+2)} \\ x_{(m+2)} + x_{(m+3)} \\ \cdot \\ \cdot \\ \cdot \\ x_{(m+n-1)} + x_{(m+n)} \\ x_{(m+n)} + x_1 \\ x_1 + x_{(m+n)} \\ x_1 + x_2 \\ x_2 + x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_{(m-1)} + x_m \end{pmatrix}.$$

Then $g_{(t+1)}(x) = T \times g_t(x)$ where T denotes the characteristic matrix of H in Proposition 3.1. And we can know that $g_t(x)$ is the vector of which the first n entries and the second m entries are the second n entries and the first m entries of $f_t(x)$, respectively, and $g_{(t+1)}(x)$ is the vector of which the first n entries and the second m entries are the second n entries and the first m entries of $f_{(t+1)}(x)$, respectively. This means that the periodicity of U is the same as the periodicity of T in Proposition 3.1. This completes the proof. □

Now we will repeat the discussion on the periodicities of some hybrid linear CA with null boundary condition which are parts of the proof of Theorem 3.3 in [5]. Let H be a hybrid CA configured with rules 60 and 102 and null boundary condition and R the rule vector of H. Then there are combined parts of different rules in the rule vector R. For each $60 - 102$ combined part in R, i.e., for each part where rule 102 just follows rule 60, we can see there is no interaction between the rules 60 and 102 in CA application. On the other hand for each $102 - 60$ combined part in R, i.e., for each part where rule 60 just follows rule 102, we can see there are interactions between the rules 102 and 60 in CA application. Now let a maximal $102 - 60$ combined part in R denote $< 102, \cdots, 102, 60, \cdots, 60 >$ which could not be extended more in R such a manner that rule 102's in a row then rule 60's in a row. And let a maximal uniform rule part in R denote $< 60, \cdots, 60 >$ or $< 102, \cdots, 102 >$ which could not be extended more in R as a uniform rule vector and could not be extended to a maximal $102 - 60$ combined part in R. Then R is a combination of maximal $102-60$ combined parts and maximal uniform rule parts of which all the parts are completely independent of each other in CA application. For each maximal $120-60$ combined part R_v in R with m_v 102's and n_v 60's, let a_v be the least non-negative integer such that $max\{m_v, n_v\} \leq 2^{a_v} + 1$. And for each maximal uniform rule part R_u of length ℓ_u in R, a_u be the non-negative integer such that $2^{a_u - 1} < \ell_u \leq 2^{a_u}$. Then we have $T^{2+2^a} = T^2$ where T is the characteristic matrix of H and a is the least common multiple of a_v's and a_u's.

We are ready to get main result.

Theorem 3.3. *Let H be a hybrid linear CA configured with rules 60 and 102 and periodic boundary condition. Let T be the characteristic matrix of H and R the rule vector of H. And let a_v's, m_v's and n_v's be as in the discussion above. Then $T^{2+2^a} = T^2$ where a takes the value as in the following:*

(1) *If the leftmost rule in R is 102 and the rightmost rule in R is 60, then a is the least common multiple of a_v's.*
(2) *If the leftmost rule in R is 60 and the rightmost rule in R is 102, then the rightmost maximal uniform part with rule 102 in R may be regarded to move to the left side of the leftmost maximal uniform part with rule 60 so that it produce a new $102-60$ combined part making a new a_v, after then, a is the least common multiple of a_v's.*

(3) *If both of the leftmost rule and the rightmost rule in* R *are* 102*, then the rightmost maximal uniform part with rule* 102 *in* R *may be regarded to move to the left side of the leftmost maximal* $102-60$ *combined part* R_v *with* m_v 102*'s so that the* m_v *increases and corresponding* a_v *could increase, after then,* a *is the least common multiple of* a_v*'s.*

(4) *If both of the leftmost rule and the rightmost rule in* R *are* 60*, then the leftmost maximal uniform part with rule* 60 *in* R *may be regarded to move to the right side of the rightmost maximal* $102-60$ *combined part* R_v *with* n_v 60*'s so that the* n_v *increases and corresponding* a_v *could increase, after then,* a *is the least common multiple of* a_v*'s.*

Proof. For (1), the characteristic matrix of H with periodic boundary condition is the same as the characteristic matrix of H with null boundary condition and there are only maximal $102-60$ combined parts in R, thus we have conclusion. For (2), as same as in the proof of Proposition 3.2, the rightmost maximal uniform part with rule 102 in R may be regarded to move to the left side of the leftmost maximal uniform part with rule 60. After then, it suffices to consider only maximal $102-60$ combined parts in R and it can be dealt with null boundary condition, thus we have the conclusion. For (3) and (4), it can be easily shown in the similar manner with (2). □

Corollary 3.4. *Let* H *be a hybrid linear CA configured with rules* 60 *and* 102 *and periodic boundary condition. Let* T *be the characteristic matrix of* H*. And let* a_v*'s be as in (1), (2), (3) and (4) of Theorem 3.3. Then* $T^{2+2^a} = T^2$ *where* a *is the maximum of* a_v*'s in each case.*

Proof. By the proof of Theorem 3.3, we can regard that the rule vector of H consists of independent maximal $102-60$ combined parts. Thus we can regard that T consists of independent submatrices corresponding to maximal $102-60$ combined parts. And $A^{2+2^q} = A^2$ for a square matrix A if $A^{2+2^p} = A^2$ and $p \leq q$ for non-negative integers p and q. So we can have the conclusion. □

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