STRICT TYPES OF REGULAR NEAR-RINGS

Yong Uk Cho (Silla University)

1. Introduction

A near-ring \( R \) is an algebraic system \( (R, +, \cdot) \) with two binary operations \( + \) and \( \cdot \) such that \( (R, +) \) is a group (not necessarily abelian) with neutral element 0, \( (R, \cdot) \) is a semigroup and \( (a+b)c = ac + bc \) for all \( a, b, c \) in \( R \). If \( R \) has a unity 1, then \( R \) is called unital. A near-ring \( R \) with the extra axiom \( a0 = 0 \) for all \( a \in R \) is said to be zero symmetric. An element \( d \) in \( R \) is called distributive if \( d(a+b) = da + db \) for all \( a \) and \( b \) in \( R \).

Mason [1] introduced the notion of left regularity and characterized left regular zero-symmetric unital near-rings. Also, several authors ([1], [2], [4] etc.) studied them.

We will use the following notations: Given a near-ring \( R \), \( R_0 = \{ a \in R \mid a0 = 0 \} \) which is called the zero symmetric part of \( R \), \( R_c = \{ a \in R \mid a0 = a \} \) which is called the constant part of \( R \).

Obviously, we see that \( R_0 \) and \( R_c \) are subnear-rings of \( R \), but \( R_d \) is a semigroup under multiplication. Clearly, near-ring \( R \) is zero symmetric, in case \( R = R_0 \) also, in case \( R = R_c \), \( R \) is called a constant near-ring.

For other notations and basic results, we shall refer to Pilz [3].

2. Results

A near-ring \( R \) is called (Von Neumann) regular if for any element \( a \in R \), there exists an element \( x \) in \( R \) such that \( a = axa \). Such an element \( a \) is called regular.

A near-ring \( R \) is said to be left regular if, for each \( a \in R \), there exists \( x \in R \) such that \( a = xa^2 \). Right regularity is defined in a symmetric way. Also, we can generalize these concepts as following.

A near-ring \( R \) is called strongly left regular if \( R \) is left regular and regular, similarly, we can define strongly right regular. A strongly left regular and strongly right regular near-ring is called strongly regular near-ring. Also, the concepts of left, strongly left, strongly right and
strong regularities are all equivalent conditions.

We say that $R$ is reduced if $R$ has no nonzero nilpotent elements, that is, for each $a$ in $R$, $a^n = 0$, for some positive integer $n$ implies $a = 0$. In ring theory, McCoy proved that $R$ is reduced if and only if for each $a$ in $R$, $a^2 = 0$ implies $a = 0$. A near-ring $R$ is said to be strongly reduced if, for $a \in R$, $a^2 \in R_c$ implies $a \in R_c$, that is, $a^2 = a^2$ implies $a0 = a$.

Obviously $R$ is strongly reduced if and only if, for $a \in R$ and any positive integer $n$, $a^n \in R_c$ implies $a \in R_c$.

Obviously, we get the following examples by the concept of strong reducibility.

**Examples 2.1.** (1) Every strongly regular near-ring is strongly reduced.
(2) Every right regular near-ring is strongly reduced.
(3) Every commutative integral near-ring is strongly reduced.

**Lemma 2.2.** Let $R$ be a strongly reduced near-ring. Then we have the following conditions.
(1) If for any $a, b \in R$ with $ab \in R_c$, then $ba \in R_c$, and $\forall x \in R$, $axb, bxa \in R_c$.

Furthermore, $ab^n \in R_c$ implies $ab \in R_c$, for each positive integer $n$.

(2) If for any $a, b \in R$ with $ab = 0$, then $ba = b0 = (ba)^2$. Moreover, $ab^n = 0$ implies $ab = 0$, for any positive integer $n$.

**Proof.** (1) Suppose that $ab \in R_c$. Then $(ba)^2 = baba = bab = bab0 \in R_c$. Since $R$ is strongly reduced, we have $ba \in R_c$.

Next, we see that $xba \in R_c$ for each $x \in R$, whence $(axb)^2 \in R_c$. By the strong reducibility of $R$, we obtain $axb \in R_c$ for each $x \in R$. Also, since $ba \in R_c$, we obtain $bxa \in R_c$ for each $x \in R$.

Furthermore, assume that $ab^n \in R_c$. Then using the first part of this (1), $(ab)^n \in R_c$. Since $R$ is strongly reduced, we see $ab \in R_c$.

(2) Assume that $ab = 0$. Then $ab \in R_c$ by (1). Hence $(ba)^2 = baba = b0 \in R_c$. Hence $ba \in R_c$. Therefore we obtain that $ba = (ba)^2 = b0$. Moreover, suppose that $ab^n = 0$. Then $ab \in R_c$ by the last part of (1), so that $ab = abb^{n-1} = ab^n = 0$. □

**Lemma 2.3.** Let $R$ be a strongly reduced near-ring. If for any $a, b \in R$ with $ab = 0$ and $a^2 = a0$, then $a = 0$. 

Proof. Suppose that for any \(a, b \in R\) with \(ab = 0\) and \(a^2 = a0\). Then \(a^2 = a0 \in R_c\). Strong reducibility implies that \(a \in R_c\). Hence we obtain that \(a = a0 = a0b = ab = 0\). \(\Box\)

Corollary 2.4. Every strongly reduced near-ring is reduced.

By Reddy and Murty [4], we say that a near-ring \(R\) has the property (*) if it satisfies the conditions:

(i) for any \(a, b \in R\), \(ab = 0\) implies \(ba = b0\).

(ii) for \(a \in R\), \(a^3 = a^2\) implies \(a^2 = a\).

Here, clearly we see that strong reducibility is equivalent to the condition (ii) and strong reducibility implies condition (i) by Lemma 2 (2).

According to the Lemmas 2.2 and 2.3, we have the following valuable corollary.

Corollary 2.5. Let \(R\) be a left (or right) regular near-ring. If for any \(a, b \in R\) with \(ab = 0\), then \((ba)^n = b0\), for all positive integer \(n\). In particular, \(ba = b0\).

Clearly, if \(R\) is a zero-symmetric near-ring, then \(R\) is strongly reduced if and only if \(R\) is reduced.

Proposition 2.6. The following statements are equivalent for a near-ring \(R\):

(1) \(R\) is strongly reduced.

(2) For \(a \in R\), \(a^3 = a^2\) implies \(a^2 = a\).

(3) If \(a^{n+1} = xa^{n+1}\) for \(x \in R\) and some nonnegative integer \(n\), then \(a = xa = ax\).

Proof (1) \(\Rightarrow\) (2). Assume that \(a^3 = a^2\). Then \((a^2 - a)a = 0\), whence \(a(a^2 - a) = a0 \in R_c\) by Lemma 2.2 (2). Then \((a^2 - a)a^2 = (a^3 - a^2)a = 0a = 0\). Again by Lemma 2.2 (2), \(a^2(a^2 - a) = a^20 \in R_c\). Hence \((a^2 - a)^2 = a^2(a^2 - a) - a(a^2 - a) = a^20 - a0 = (a^2 - a)0 \in R_c\). This implies \(a^2 - a = 0\). Hence \(a^2 - a = (a^2 - a)0 = (a^2 - a)a = 0\).

(2) \(\Rightarrow\) (1). Assume \(a^2 \in R_c\). Then \(a^3 = a^2a = a^2\). By hypothesis, this implies \(a = a' \in R_c\).

(1) \(\Rightarrow\) (3). Suppose \(a^{n+1} = xa^{n+1}\) for some \(n \geq 0\). Then \((a - xa)a^n = 0\). Hence \((a - xa)a = 0\) by Lemma 2.2 (2), and so \((a - xa)^2 \in R_c\) by Lemma 2.2 (1). Since \(R\) is strongly reduced, we have \(a - xa \in R_c\). Then \(a - xa = (a - xa)a = 0\), that is \(a = xa\). Now
(a - ax)a = a^2 - axa = a^2 - a^2 = 0 \in R_e. Hence (a - ax)^2 = a(a - ax) - ax(a - ax) \in R_e by Lemma 2.2 (1), and so a - ax \in R_e. Therefore a - ax = (a - ax)a = 0.

(3) \implies (2). This is obvious. □

References


