2-TYPE SURFACES AND QUADRIC HYPERSURFACES SATISFYING $\langle \Delta x, x \rangle = \text{const.}$

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Abstract. Let $M$ be a connected $n$-dimensional submanifold of a Euclidean space $E^{n+k}$ equipped with the induced metric and $\Delta$ its Laplacian. If the position vector $x$ of $M$ is decomposed as a sum of three vectors $x = x_1 + x_2 + x_0$ where two vectors $x_1$ and $x_2$ are non-constant eigenvectors of the Laplacian, and $x_0$ is a constant vector, then, $M$ is called a 2-type submanifold. In this paper we showed that a 2-type surface $M$ in $E^3$ satisfies $\langle \Delta x, x - x_0 \rangle = c$ for a constant $c$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in $E^3$, then $M$ is an open part of a circular cylinder. Also we showed that if a quadric hypersurface $M$ in a Euclidean space satisfies $\langle \Delta x, x \rangle = c$ for a constant $c$, then it is one of a minimal quadric hypersurface, a general cone, a hypersphere, and a spherical cylinder.

1. Introduction

Let $M$ be an $n$-dimensional submanifold of the $(n+k)$-dimensional Euclidean space $E^{n+k}$, equipped with the induced metric. Denote by $\Delta$ the Laplacian of $M$. If the position vector $x$ of $M$ in $E^{n+k}$ can be decomposed as a finite sum of non-constant eigenvectors of $\Delta$, we shall say that $M$ is of finite-type. More precisely, $M$ is said to be of $q$-type if the position vector $x$ of $M$ can be expressed as in the following form:

$$x = x_0 + x_{i_1} + \cdots + x_{i_q},$$

where $x_0$ is a constant vector, and $x_{i_j}$ ($j = 1, \cdots, q$) are non-constant vectors in $E^{n+k}$ such that $\Delta x_{i_j} = \lambda_{i_j} x_{i_j}$, $\lambda_{i_j} \in R$, $\lambda_{i_1} < \cdots < \lambda_{i_q}$. The notion of finite-type submanifolds has been introduced by B.-Y. Chen [1]. Many results concerning this subject are obtained during last three decades. One of the interesting research areas on this subject is a classification of 2-type submanifolds. Th.Hasanis and Th.Vlachos proved that the only 2-type surface in the three dimensional sphere $S^3$ is an open part of a product of two circles of different radii [4]. Also they proved that a spherical hypersurface $M$ is of 2-type if and only if it
has constant scalar curvature and mean curvature [5]. In [2] B.-Y. Chen studied a special 2-type surface $M$ in $E^3$ whose position vector $x$ can be decomposed as a sum of two non-constant eigenvectors $x = x_1 + x_2$, $\Delta x_1 = 0$, $\Delta x_2 = \lambda x_2$, $0 \neq \lambda \in \mathbb{R}$. Such a 2-type surface is said to be of null 2-type. Especially he proved that the only null 2-type surface in $E^3$ is a circular cylinder. Many studies on null 2-type submanifolds are followed. But until now generally 2-type surfaces are not classified. We can notice that every known finite-type hypersurface $M$ satisfies the condition $\langle \Delta x, x \rangle = c$ for a constant $c$, where $x$ is the position vector of $M$ and $\langle \ , \ \rangle$ denotes the usual inner product in Euclidean space. Note that the condition $\langle \Delta x, x \rangle = c$ for a constant $c$ is not coordinate invariant. Sometimes a parallel translation is necessary to see that this condition can be satisfied. So we would like to study finite-type submanifold satisfying the condition $\langle \Delta x, x \rangle = c$ for a constant $c$. In Section 3 we will show that if a 2-type surface $M$ in $E^3$ satisfies the condition $\langle \Delta x, x - x_0 \rangle = c$ for a constant $c$, then it is an open part of a circular cylinder. In [3] B.-Y. Chen, F. Dillen and H. Z. Song proved that if $M$ is a quadric hypersurface of finite-type in a Euclidean space, then $M$ is one of a minimal quadric hypersurface, a spherical cylinder, and a hypersphere. In Section 4, we will show that if a quadric hypersurface $M$ in a Euclidean space satisfies the condition $\langle \Delta x, x \rangle = c$ for a constant $c$, then it is one of a minimal quadric hypersurface, a generalized cone, a hypersphere, and a spherical cylinder.

2. Preliminaries

Consider an $n$-dimensional submanifold $M$ of $E^{n+1}$ and denote $\nabla$ and $\bar{\nabla}$ the usual Riemannian connection of $E^{n+1}$ and the induced connection on $M$, respectively. The formulas of Gauss and Weingarten are given respectively by

\begin{align*}
\bar{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\
\bar{\nabla}_X \xi &= -A_\xi X + D_X \xi
\end{align*}

for vector fields $X, Y$ tangent to $M$ and $\xi$ normal to $M$, where $h$ is the second fundamental form, $D$ the normal connection, and $A$ the shape operator of $M$. For each normal vector $\xi$ at a point $p \in M$, the shape operator $A_\xi$ is a self-adjoint operator of the tangent space $T_p M$ at $p$. The second fundamental form $h$ and the shape operator $A$ are related by

$$\langle A_\xi X, Y \rangle = \langle h(X, Y), \xi \rangle,$$

where $\langle \ , \ \rangle$ is the usual inner product in $E^{n+1}$. Let $v$ be an $E^{n+1}$-valued smooth function on $M$, and let $\{e_1, e_2, \cdots, e_n\}$ be a local orthonormal frame field of $M$. We define

$$\Delta v = \sum_{i=1}^{n} (\bar{\nabla}_{e_i} \nabla_{e_i} v - \bar{\nabla}_{\nabla_{e_i} e_i} v).$$
It is well known that the position vector $x$ and the mean curvature vector $H$ of $M$ in $E^{n+1}$ satisfy

$$\Delta x = H. \quad (4)$$

Let $e_{n+1}$ be a local unit normal vector to $M$. Since the mean curvature vector $H$ is normal to $M$, we have $H = \langle H, e_{n+1} \rangle e_{n+1}$. The function $\langle H, e_{n+1} \rangle$ is called mean curvature function and it will be denoted by $\alpha$.

### 3. 2-type surface in $E^3$ satisfying $\langle \Delta x, x - x_0 \rangle = \text{const.}$

Let $M$ be a 2-type surface in $E^3$. Then its position vector $x$ is expressed in the form

$$x = x_0 + x_1 + x_2,$$

where $x_0$ is a constant vector, and $x_i (i = 1, 2)$ are nonconstant vectors in $E^3$ such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in R$, $\lambda_1 \neq \lambda_2$. By (4) we have $\Delta x = H = \lambda_1 x_1 + \lambda_2 x_2$ and $\Delta^2 x = \Delta H = \lambda_1^2 x_1 + \lambda_2^2 x_2$. Thus

$$\Delta^2 x = (\lambda_1 + \lambda_2) \Delta x - \lambda_1 \lambda_2 (x - x_0). \quad (5)$$

The general basic formula of $\Delta H$ derived in [1] plays important role in the study of low type. In particular, if $M$ is a surface in $E^3$, it reduces to

$$\Delta H = (\Delta \alpha - \alpha ||A_{e_3}||^2) e_3 - 2\alpha A_{e_3} (\text{grad} \alpha) - \alpha \text{grad} \alpha, \quad (6)$$

where $\alpha$ is the mean curvature function and $e_3$ a unit normal vector of $M$ in $E^3$. By comparing the tangential part of both (5) and (6), we find

$$\lambda_1 \lambda_2 (x - x_0)^T = 2A_{e_3} (\text{grad} \alpha) + \alpha \text{grad} \alpha, \quad (7)$$

where $(x - x_0)^T$ means the tangential part of the vector $x - x_0$. Now suppose that

$$\langle \Delta x, x - x_0 \rangle = c \quad (8)$$
holds for a constant $c$. Let $\{e_1, e_2\}$ be a local orthonormal frame of $M$. Since

$$\Delta \langle \Delta x, x - x_0 \rangle = \sum_{i=1}^{2} e_i e_i \langle \Delta x, x - x_0 \rangle - \sum_{i=1}^{2} \nabla e_i e_i \langle \Delta x, x - x_0 \rangle$$

$$= \sum_{i=1}^{2} e_i \left( \langle \nabla e_i (\Delta x), x - x_0 \rangle + \langle \Delta x, e_i \rangle \right)$$

$$- \sum_{i=1}^{2} \left( \langle \nabla \nabla e_i e_i (\Delta x), x - x_0 \rangle + \langle \Delta x, \nabla e_i e_i \rangle \right)$$

$$= \sum_{i=1}^{2} \langle \nabla e_i (\Delta x), x - x_0 \rangle - \sum_{i=1}^{2} \langle \nabla \nabla e_i e_i (\Delta x), x - x_0 \rangle$$

$$- \sum_{i=1}^{2} \langle \nabla \nabla e_i e_i (\Delta x), x - x_0 \rangle$$

$$= \langle \Delta (\Delta x), x - x_0 \rangle + \sum_{i=1}^{2} \langle \nabla e_i (\Delta x), e_i \rangle$$

$$= \langle \Delta^2 x, x - x_0 \rangle + \sum_{i=1}^{2} \langle D e_i (\Delta x) - A_{\Delta x} e_i, e_i \rangle \ (\text{by (2)})$$

$$= \langle \Delta^2 x, x - x_0 \rangle - \sum_{i=1}^{2} \langle A_{\Delta x} e_i, e_i \rangle$$

$$= \langle \Delta^2 x, x - x_0 \rangle - \sum_{i=1}^{2} \langle \Delta x, h(e_i, e_i) \rangle \ (\text{by (3)})$$

$$= \langle \Delta^2 x, x - x_0 \rangle - \langle \Delta x, \Delta x \rangle,$$

(8) and $\Delta x = H = \alpha e_3$ imply

$$\langle \Delta^2 x, x - x_0 \rangle - \alpha^2 = 0. \quad (9)$$

From (5), (8) and (9), we get

$$(\lambda_1 + \lambda_2) c - \lambda_1 \lambda_2 \langle x - x_0, x - x_0 \rangle - \alpha^2 = 0.$$  

Differentiating both sides of the above equation in the direction of a tangent vector $X$ on $M$, we find

$$-2\lambda_1 \lambda_2 \langle x - x_0, X \rangle - 2\alpha X(\alpha) = 0$$

or

$$X(\alpha) = -\frac{\lambda_1 \lambda_2}{\alpha} \langle X, (x - x_0)^T \rangle.$$
This implies that
\[ \text{grada} = -\frac{\lambda_1\lambda_2}{\alpha}(x - x_0)^T. \] (10)

**Lemma 3.1.** Let \( M \) be a 2-type surface in \( E^3 \) whose position vector \( x \) is expressed as \( x = x_0 + x_1 + x_2 \), where \( x_0 \) is a constant vector, and \( x_i (i = 1, 2) \) are nonconstant vectors in \( E^3 \) such that \( \Delta x_i = \lambda_i x_i, \lambda_i \in \mathbb{R}, \lambda_1 \neq \lambda_2 \). Assume that \( \langle \Delta x, x - x_0 \rangle = c \) holds for a constant \( c \). Then the mean curvature function \( \alpha \) of \( M \) is constant.

**Proof.** Suppose that \( \alpha \) is nonconstant. If \( M \) is of null 2-type, then \( M \) is a circular cylinder \([2]\), which implies that the mean curvature function \( \alpha \) is constant. So the assumption implies that \( M \) is not of null 2-type. Substituting (10) into (7) we get
\[ A_{e_3}(x - x_0)^T = -\alpha(x - x_0)^T, \]
which implies that \( \text{grada} \) is a principal vector of the shape operator \( A_{e_3} \) and the corresponding principal curvature is \( -\alpha \). Since \( \alpha \) is the sum of two principal curvatures, the other principal curvature is \( 2\alpha \). Let \( \{e_1, e_2\} \) be a local orthonormal frame of \( M \) such that \( e_1 \) is parallel to \( \text{grada} \). Note that \( e_2(\alpha) = 0 \). By the Coddazzi equations, we have
\[ e_1(2\alpha) = (-\alpha - 2\alpha)\omega_{12}(e_2) = -3\alpha\omega_{12}(e_2), \] (11)
\[ e_2(-\alpha) = (-\alpha - 2\alpha)\omega_{12}(e_1) = -3\alpha\omega_{12}(e_1), \] (12)
where \( \omega_{12} \) is the connection form of \( \{e_1, e_2\} \). Since \( \alpha \) is nonzero and \( e_2(\alpha) = 0 \), from (12) it follows that \( \omega_{12}(e_1) = 0 \). From (11) we have \( \omega_{12}(e_2) = \frac{2e_1(\alpha)}{3\alpha} \).

This and \( \omega_{12}(e_1) = 0 \) implies that
\[ \omega_{12} = -\frac{2e_1(\alpha)}{3\alpha} \theta_2, \] (13)
where \( \{\theta_1, \theta_2\} \) denotes the dual 1-forms of \( \{e_1, e_2\} \). Since \( \text{grada} = e_1(\alpha)e_1 \), by (10) we find
\[ \langle x - x_0, e_2 \rangle = 0. \]
Differentiating both sides of the above in the direction of \( e_2 \), we find
\[ 1 + \langle x - x_0, \nabla e_2 e_2 \rangle = 0. \] (14)

By (1) and \( h(e_2, e_2) = 2\alpha e_3 \) we have
\[ \nabla e_2 e_2 = h(e_2, e_2) + \nabla e_2 e_2 = 2\alpha e_3 + \omega_{21}(e_2)e_1. \]
Substituting this into (14) and we find
\[ 1 + 2\langle x - x_0, \alpha e_3 \rangle + \omega_{21}(e_2)\langle x - x_0, e_1 \rangle = 0. \]
By using (8), (10), (13) and considering \( \text{grada} = e_1(\alpha)e_1 \) it follows that
\[ 1 + 2c - \frac{2(e_1(\alpha))^2}{3\lambda_1\lambda_2} = 0. \]
from the above equation. This implies that \( e_1(\alpha) \) is a constant. Since \( d\omega_{12} = -K\theta_1 \wedge \theta_2 \), where \( K \) is the Gauss curvature of \( M \), from (13) and the structural equation \( d\theta_2 = \omega_{21} \wedge \theta_1 \), we get

\[
-K\theta_1 \wedge \theta_2 = \frac{-2e_1(\alpha)}{3}(-\frac{e_1(\alpha)}{\alpha^2}\theta_1 \wedge \theta_2) - \frac{2e_1(\alpha)}{3\alpha}(-\frac{2e_1(\alpha)}{3\alpha}\theta_1 \wedge \theta_2)
\]

\[
= \frac{10e_1(\alpha)^2}{9\alpha^2}\theta_1 \wedge \theta_2.
\]

Since \( K = -2\alpha^2 \), from this we have \( 18\alpha^4 = 10(e_1(\alpha))^2 \), which implies that \( \alpha \) is constant. This is a contradiction. □

Proposition 3.2. Let \( M \) be a 2-type surface in \( E^3 \) whose position vector \( x \) is expressed as \( x = x_0 + x_1 + x_2 \), where \( x_0 \) is a constant vector, and \( x_i (i = 1, 2) \) are nonconstant vectors in \( E^3 \) such that \( \Delta x_i = \lambda_i x_i, \lambda_i \in R, \lambda_1 \neq \lambda_2 \). Assume that \( \langle \Delta x, x - x_0 \rangle = c \) holds for a constant \( c \). Then \( M \) is of null 2-type, i.e., \( M \) is an open part of a circular cylinder.

Proof. By Lemma 3.1, the mean curvature function \( \alpha \) of \( M \) is constant. By (6) it implies that \( \Delta^2 x = \Delta H \) is normal to \( M \). From (5) it follows that \( \lambda_1 \lambda_2 (x - x_0) \) is normal to \( M \). If \( M \) is not of null 2-type, then the vector \( x - x_0 \) is normal to \( M \). This is impossible. Thus \( M \) is of null 2-type. Consequently \( M \) is an open part of a circular cylinder [2]. □

4. Quadric hypersurfaces satisfying \( \langle \Delta x, x \rangle = \text{const.} \)

Consider the set \( M \) of points \( (x_1, \ldots, x_{n+1}) \) in the \((n + 1)\)-dimensional Euclidean space \( E^{n+1} \) satisfying the following equation of the second degree:

\[
\sum_{i,j=1}^{n+1} a_{ij}x_ix_j + \sum_{i=1}^{n+1} b_ix_i + d = 0,
\]

(15)

where \( a_{ij}, b_j, d \) are real numbers. The equation can be expressed as in the following form

\[
\langle Ax + b, x \rangle + d = 0,
\]

where \( \langle \ , \ \rangle \) is the usual inner product of \( E^{n+1} \), for the matrix \( A = (a_{ij}) \) and vectors \( x = \begin{bmatrix} x_1 \\ \vdots \\ x_{n+1} \end{bmatrix} \), \( b = \begin{bmatrix} b_1 \\ \vdots \\ b_{n+1} \end{bmatrix} \). We can assume without loss of generality that the matrix \( A = (a_{ij}) \) is symmetric and \( A \) is not a zero matrix. If the left side of the equation (15) is reducible polynomial, then \( M \) is a hyperplane or a union of two hyperplanes. In this paper we assume that the polynomial given by the left side of (15) is irreducible over real numbers. In general the whole set \( M \) does not form a submanifold of \( E^{n+1} \). Instead it can be shown that the subset

\[
M' = \{ x = \begin{bmatrix} x_1 \\ \vdots \\ x_{n+1} \end{bmatrix} \in M|2Ax + b \neq 0 \} \text{ is an } n\text{-dimensional submanifold}
\]
of $E^{n+1}$ by using the implicit function theorem. In this paper, we mean the hypersurface $M'$ by a quadric hypersurface $M$ described by (15). We will study a quadric hypersurface $M$ satisfying the condition $\langle \Delta x, x \rangle = c$ for a constant $c$, where $x$ is the position vector of $M$ and $\Delta$ its Laplacian. Note that the condition $\langle \Delta x, x \rangle = c$ for a constant $c$ is invariant under an orthogonal transformation. So without loss of generality we may assume that the matrix $A$ is diagonal with diagonal entries $\lambda_1, \cdots, \lambda_{n+1}$. So the equation (15) can be written as

$$\sum_{i=1}^{n+1} \lambda_i x_i^2 + \sum_{i=1}^{n+1} b_i x_i + d = 0, \quad (16)$$

or

$$\langle Ax + b, x \rangle + d = 0, \quad (17)$$

where $A$ is the diagonal matrix $\text{diag}[\lambda_1, \cdots, \lambda_{n+1}]$. Note again that we only consider the case that the left side of (16) is irreducible. First of all we will investigate some basic properties of quadric hypersurface $M$ and classify the minimal quadric hypersurfaces in an elementary way.

**Lemma 4.1.** The vector $2Ax + b$ is a nonzero normal vector to $M$.

**Proof.** Differentiating both sides of (17) in the direction of a tangent vector field $X$ of $M$, we find

$$\langle AX, x \rangle + \langle Ax + b, X \rangle = 0$$

or

$$\langle 2Ax + b, X \rangle = 0.$$  

This implies that $2Ax + b$ is normal to $M$. By assumption $2Ax + b$ is nonzero. \qed

**Lemma 4.2.** Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame of $M$. Then the following holds.

$$\sum_{i=1}^{n} \langle 2Ae_i, e_i \rangle + \langle Ax + b, \Delta x \rangle = 0. \quad (18)$$

**Proof.** Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame of $M$. By Lemma 4.1 we have

$$\langle 2Ax + b, e_i \rangle = 0$$

for $i = 1, 2, \cdots, n$. Differentiating the above equation in the direction of $e_i$ we find

$$\langle 2Ae_i, e_i \rangle + \langle 2Ax + b, h(e_i, e_i) \rangle = 0,$$

where $h$ is the second fundamental form of $M$. Since $\Delta x = \sum_{i=1}^{n} h(e_i, e_i)$, by summing up over $i$ we get (18). \qed

It is already well-known that the only minimal quadric hypersurfaces are cones described in the following lemma. But we will prove it by using Lemma 4.2.
Lemma 4.3. If $M$ is a minimal quadric hypersurface, then by a parallel translation and an orthogonal coordinate change, it can be described by

$$(l-1)\sum_{i=1}^{k} x_i^2 + (1-k) \sum_{i=k+1}^{k+l} x_i^2 = 0$$

for integers $k, l (k, l > 1, k + l \leq n + 1)$.

Proof. Let $M$ be a minimal quadric hypersurfaces described by (16). Since the condition minimality is invariant under any parallel translation and orthogonal coordinate change, we may write the equation (16) as

$$s \sum_{i=1}^{s} \lambda_i x_i^2 + \sum_{i=s+1}^{s+t} b_i x_i + d = 0 \quad (\lambda_i \neq 0, \ i = 1, \cdots, s)$$

or

$$s \sum_{i=1}^{s} \lambda_i x_i^2 + \sum_{i=s+1}^{s+t} b_i x_i + d = 0.$$

( $\lambda_i \neq 0, \ i = 1, \cdots, s, \ b_j \neq 0, \ j = s + 1, \cdots, s + t \leq n + 1$)

We will show that the second description is impossible. Suppose that $M$ is described by the second equation. Let $e_1, \cdots, e_n$ be a local orthonormal frame of $M$. Since $2Ax + b$ is a normal vector field of $M$. Thus $e_1, \cdots, e_n$ and $\frac{2Ax + b}{|2Ax + b|}$ form a Euclidean orthonormal frame, where $|2Ax + b|$ means the magnitude of the vector $2Ax + b$. So we have

$$\sum_{i=1}^{n} \langle 2A e_i, e_i \rangle + \langle 2A \frac{2Ax + b}{|2Ax + b|}, \frac{2Ax + b}{|2Ax + b|} \rangle = \text{tr}(2A),$$

where $\text{tr}(2A)$ is the trace of the matrix $2A$. Since $M$ is minimal, it follows from (18) and the above equation that

$$\langle 2A(2Ax + b), 2Ax + b \rangle = \text{tr}(2A) \langle 2Ax + b, 2Ax + b \rangle.$$  (19)

Since $b_{s+1} \neq 0$, $M$ can be locally considered as a graph of the function $x_{s+1} = \frac{1}{b_{s+1}} (-d - \sum_{i=1}^{s} \lambda_i x_i^2 - \sum_{i=s+2}^{s+t} b_i x_i)$. The equation (19) can be written as

$$\sum_{i=1}^{s} 4\lambda_i^2 (\text{tr}(2A) - 2\lambda_i)x_i^2 + \sum_{i=s+1}^{s+t} b_i^2 = 0.$$

As $x_1, \cdots, x_s$ are independent variables, from the above equation, we have $\lambda_i = \text{tr}(A), \ i = 1, \cdots, s$ and $\text{tr}(2A) \sum_{i=s+1}^{s+t} b_i^2 = 0$. From this we find $\lambda_i = 0, \ i = 1, \cdots, s$, which is a contradiction. Thus we know that $b = 0$, which implies $\langle Ax, x \rangle + d = 0$, or $\sum_{i=1}^{s} \lambda_i x_i^2 + d = 0$. The equation (19) can be simplified as

$$\langle A^2 x, Ax \rangle = \text{tr}(A) \langle Ax, Ax \rangle.$$  (20)
Without loss of generality we may consider $M$ as a graph of the function $x_1 = \pm \frac{1}{\sqrt{d}} \sqrt{-d - \sum_{i=2}^{s} \lambda_i x_i^2}$. Substituting this into (20) we get

$$\sum_{i=2}^{s} \lambda_i (\lambda_i^2 - \operatorname{tr}(A) \lambda_i - \lambda_1^2 + \operatorname{tr}(A) \lambda_1) x_i^2 - \lambda_1 d(\lambda_1 - \operatorname{tr}(A)) = 0.$$ 

From this we have

$$\lambda_i^2 - \operatorname{tr}(A) \lambda_i - \lambda_1^2 + \operatorname{tr}(A) \lambda_1 = 0, \quad i = 2, \ldots, s, \quad \lambda_1 d(\lambda_1 - \operatorname{tr}(A)) = 0.$$ 

From the second equation, we have $d = 0$ or $\lambda_1 = \operatorname{tr}(A)$. If $\lambda_1 = \operatorname{tr}(A)$, then the first equation and the condition $\lambda_i \neq 0, i = 2, \ldots, s$ we find $\lambda_i = \operatorname{tr}(A), i = 1, \ldots, s$, which implies that $s = 1$ or $M$ and thus $\lambda_1 x_1^2 + d$ is reducible. So we have $d = 0$. The first equation is factorized into

$$(\lambda_i - \lambda_1)(\lambda_i - (\operatorname{tr}(A) - \lambda_1)) = 0,$$

which implies that $\lambda_i = \lambda_1$ or $\lambda_i = \operatorname{tr}(A) - \lambda_1, i = 1, \ldots, s$. If all $\lambda_i = \lambda_1$, then $\lambda \sum_{i=1}^{s} x_i^2 = 0$ or $x_1 = \cdots = x_s = 0$, which is impossible. So without loss of generality, we may assume that $\lambda_1 = \cdots = \lambda_k$ and $\lambda_{k+1} = \cdots = \lambda_s$ for some positive integer $k, 1 \leq k < s$. Suppose that $k = 1$. Then, since $\operatorname{tr}(A) = \lambda_1 + (s-1)(\operatorname{tr}(A) - \lambda_1), (s-2)(\operatorname{tr}(A) - \lambda_1) = 0$. This implies that $s = 2$ or $\operatorname{tr}(A) - \lambda_1 = 0$. In any cases, the polynomial $\sum_{i=1}^{s} \lambda_i x_i^2$ is reducible. So we may assume that $1 < k < s-1$. Let $\lambda_1 = \lambda, \operatorname{tr}(A) - \lambda_1 = \mu$ and $s - k = l$. From $\operatorname{tr}(A) = k\lambda + l\mu$ and $\mu = \operatorname{tr}(A) - \lambda$, we have $\mu = \frac{k-1}{k+l}\lambda$. So given quadric hypersurface can be described as

$$\lambda \sum_{i=1}^{k} x_i^2 + \frac{1-k}{l-1} \lambda \sum_{i=k+1}^{k+l} x_i^2 = 0$$

or

$$(l-1) \sum_{i=1}^{k} x_i^2 + (1-k) \sum_{i=k+1}^{k+l} x_i^2 = 0$$

(21)

for some two positive integers $k, l > 1, k + l \leq n + 1$. Conversely, we can show that a quadric hypersurface described by (21) is a minimal hypersurface. Let $M$ be a quadric hypersurface in $E^{n+1}$ described by (21). The equation (21) can be written as $\langle A x, x \rangle = 0$, where $A$ is an $(n+1) \times (n+1)$ diagonal matrix with diagonals $l-1, \ldots, l-1, l-1, \cdots, k, 0, \cdots, 0$. Let $e_1, \ldots, e_n$ be a local orthonormal frame of $M$. Since $\frac{Ax}{|Ax|}$ is a unit normal vector to $M$, we have

$$\langle Ae_1, e_1 \rangle + \cdots + \langle Ae_n, e_n \rangle + \langle A \frac{Ax}{|Ax|}, \frac{Ax}{|Ax|} \rangle = \operatorname{tr}(A) = k(l-1) + l(1-k) = l - k.$$

(22)
By using (21) we have
\[
\langle A \frac{Ax}{|Ax|}, Ax \rangle = \frac{(l - 1)^3 \sum_{i=1}^{k} x_i^2 + (1 - k)^3 \sum_{i=k+1}^{k+l} x_i^2}{(l - 1)^2 \sum_{i=1}^{k} x_i^2 + (1 - k)^2 \sum_{i=k+1}^{k+l} x_i^2}
\]
\[
= \frac{(l - 1)^3 \sum_{i=1}^{k} x_i^2 + (1 - l)(1 - k)^2 \sum_{i=1}^{k} x_i^2}{(l - 1)^2 \sum_{i=1}^{k} x_i^2 + (1 - l)(1 - k) \sum_{i=1}^{k} x_i^2}
\]
\[
= l - k.
\]
So from (22) and the above equation we get
\[
\langle Ae_1, e_1 \rangle + \cdots + \langle Ae_n, e_n \rangle = 0. \tag{23}
\]
By similar computation in Lemma 4.2, we have
\[
\langle Ae_1, e_1 \rangle + \cdots + \langle Ae_n, e_n \rangle + \langle Ax, \Delta x \rangle = 0.
\]
This and (23) imply that \(\langle Ax, \Delta x \rangle = 0\). Subsequently we have \(\Delta x = 0\). So we can conclude that \(M\) is minimal. □

From now on we assume that \(M\) is a quadric hypersurface described by \(\langle Ax + b, x \rangle + d = 0\) for an \((n + 1) \times (n + 1)\) diagonal matrix \(A\) with diagonal entries \(\lambda_1, \ldots, \lambda_{n+1}\) and a constant vector \(b = \begin{bmatrix} b_1 \\ \vdots \\ b_{n+1} \end{bmatrix}\) in \(E^{n+1}\) and satisfies \(\langle \Delta x, x \rangle = c\) for a constant \(c\).

**Lemma 4.4.** Assume that \(c \neq 0\). Then the following holds.

\[
\text{tr}(2A)\langle 2Ax + b, 2Ax + b \rangle \langle 2Ax + b, x \rangle - \langle 2A(2Ax + b), 2Ax + b \rangle \langle 2Ax + b, x \rangle \\
+ c(2Ax + b, 2Ax + b)^2 = 0.
\]

**Proof.** Let \(\{e_1, \ldots, e_n\}\) be a local orthonormal frame of \(M\). Then by Lemma 4.2 the following holds.
\[
\sum_{i=1}^{n} \langle 2Ae_i, e_i \rangle + \langle 2Ax + b, \Delta x \rangle = 0. \tag{24}
\]

Also we have
\[
\sum_{i=1}^{n} \langle 2Ae_i, e_i \rangle + \langle 2A \frac{2Ax + b}{|2Ax + b|}, \frac{2Ax + b}{|2Ax + b|} \rangle = \text{tr}(2A). \tag{25}
\]
Since both \(2Ax + b\) and \(\Delta x\) are normal to \(M\), there exists a scalar function \(f(x)\) defined on \(M\) such that \(\Delta x = f(x)(2Ax + b)\). This and (24) imply that \(\sum_{i=1}^{n} \langle 2Ae_i, e_i \rangle = -f(x)(2Ax + b, 2Ax + b)\). Substituting this into (25), we have
\[
\text{tr}(2A) - \frac{\langle 2A(2Ax + b), 2Ax + b \rangle}{(2Ax + b, 2Ax + b)} + f(x)(2Ax + b, 2Ax + b) = 0.
\]
From this and \( \langle \Delta x, x \rangle = f(x)\langle 2Ax + b, x \rangle = c \), it follows that
\[
\text{tr}(2A) - \frac{\langle 2A(2Ax + b), 2Ax + b \rangle}{\langle 2Ax + b, 2Ax + b \rangle} + \frac{c}{\langle 2Ax + b, x \rangle} \langle 2Ax + b, 2Ax + b \rangle = 0
\]
or
\[
\text{tr}(2A)\langle 2Ax + b, 2Ax + b \rangle\langle 2Ax + b, x \rangle - \langle 2A(2Ax + b), 2Ax + b \rangle\langle 2Ax + b, x \rangle + c\langle 2Ax + b, 2Ax + b \rangle^2 = 0.
\]
\( \square \)

We proceed two cases separately.

Case 1. \( \langle \Delta x, x \rangle = 0 \)

If \( \Delta x = 0 \), then \( M \) is a minimal hypersurface. Assume that \( M \) is nonminimal, that is, \( \Delta x \neq 0 \). As both of \( \Delta x \) and \( 2Ax + b \) are normal to \( M \), there exists a nonzero scalar function \( f(x) \) defined on \( M \) such that \( \Delta x = f(x)(2Ax + b) \).

From \( 0 = \langle \Delta x, x \rangle = f(x)\langle 2Ax + b, x \rangle \), we get \( \langle 2Ax + b, x \rangle = 0 \) . From this and \( \langle Ax + b, x \rangle + d = 0 \), we have \( \langle Ax, x \rangle = d \). We can deduce that \( Ax \) is a normal vector field of \( Ax + b \). Since \( 2Ax + b \) is also normal, we can see that if \( b \) is non-zero vector, then \( b \) is a constant normal vector of \( M \). As \( M \) is not a hyperplane, it is impossible. So we can say that \( b = 0 \) and consequently \( \langle Ax, x \rangle = 0 \). Therefore we can conclude that a quadric hypersurface satisfies \( \Delta x = 0 \), then \( M \) is a minimal quadric hypersurface described in Lemma 4.3 or a nonminimal quadric hypersurface described by \( \langle Ax, x \rangle = 0 \) for a diagonal matrix \( A \).

Case 2. \( \langle \Delta x, x \rangle = c \neq 0 \)

First we will show that if \( \lambda_i = 0 \), then \( b_i = 0 \) for \( i \in \{1, \cdots, n + 1\} \). Suppose that \( \lambda_1 = 0 \) and \( b_1 \neq 0 \). Then \( M \) can be locally considered a graph of function \( x_1 = \frac{1}{b_1}(-d - \sum_{i=2}^{n+1} \lambda_i x_i^2 - \sum_{i=2}^{n+1} b_i x_i) \), since \( \langle Ax + b, x \rangle = d \). Lemma 4.2 and \( \langle Ax + b, x \rangle + d = 0 \) imply that
\[
\text{tr}(2A)\langle 2Ax + b, 2Ax + b \rangle \langle Ax, x \rangle - \langle 2A(2Ax + b), 2Ax + b \rangle \langle Ax, x \rangle - \langle 2A(2Ax + b), 2Ax + b \rangle \langle Ax, x \rangle - \langle 2A(2Ax + b), 2Ax + b \rangle \langle Ax, x \rangle \\
+c(\langle 2Ax + b, 2Ax + b \rangle)^2 = 0.
\]

We can observe the left side of (26) is a polynomial of \( x_2, \cdots, x_{n+1} \), which are independent variables. So it must be identically zero. If we consider the coefficients of the term \( x_i^4, i = 2, \cdots, n + 1 \) of this polynomial, we find
\[
4\text{tr}(2A)\lambda_i^3 - 8\lambda_i^4 + 16c\lambda_i^4 = 0, \; i = 2, \cdots, n + 1.
\]
This implies that
\[
\lambda_i = 0 \text{ or } (2 - 4c)\lambda_i = \text{tr}(2A), \; i = 2, \cdots, n + 1. \tag{27}
\]
Now consider the coefficients of \( x_i^2 x_j^2 (2 \leq i, j \leq n + 1, i \neq j) \). Then we find
\[
4\text{tr}(2A)(\lambda^2 \lambda_j + \lambda^2 \lambda_i) - 8(\lambda_i^3 \lambda_j + \lambda_j^3 \lambda_i) + 32c\lambda_i^2 \lambda_j^2 = 0. \tag{28}
\]
If \(2 - 4c = 0\), then from (27) we find \(\text{tr}(2A) = 0\). This and (28) imply that all \(\lambda_i\) are equally zero. It’s a contradiction. So we can see that \(2 - 4c \neq 0\). Consequently from (27) we may assume that
\[
\lambda_i = \lambda \neq 0, \ i = 2, \cdots, k
\]
and
\[
\lambda_i = 0, \ i = k + 1, \cdots, n + 1.
\]
So the equation (26) can be written as
\[
\text{tr}(2A)(4\lambda^2 \sum_{i=2}^{k} x_i^2 + 4\lambda \sum_{i=2}^{k} b_i x_i + \sum_{i=1}^{n+1} b_i^2)(\lambda \sum_{i=2}^{k} x_i^2 - d) = 0
\]
\[
(4\lambda^2 \sum_{i=2}^{k} x_i^2 + 8\lambda^2 \sum_{i=2}^{k} b_i x_i + 2\lambda \sum_{i=2}^{k} b_i^2)(\lambda \sum_{i=2}^{k} x_i^2 - d) + c(4\lambda^2 \sum_{i=2}^{k} x_i^2 + 4\lambda \sum_{i=2}^{k} b_i x_i + \sum_{i=1}^{n+1} b_i^2)^2 = 0. \tag{29}
\]

If we consider the coefficient of the term \(x_2 x_i\) \((i = 3, \cdots, k)\) of the left side of (29), it is equal to \(32c\lambda b_2 b_i\), which must be zero. Suppose that \(b_2 \neq 0\). It follows that \(b_i = 0, \ i = 3, \cdots, k\). So the coefficients of the terms \(x_3^2\) and \(x_2^2\) are equals to
\[
-4d\lambda^2 \text{tr}(2A) + \text{tr}(2A)\lambda(\sum_{i=1}^{n+1} b_i^2) + 8d\lambda^3 - 2\lambda^2 b_2^2 + 8c\lambda^2(\sum_{i=1}^{n+1} b_i^2)
\]
and
\[
-4d\lambda^2 \text{tr}(2A) + \text{tr}(2A)\lambda(\sum_{i=1}^{n+1} b_i^2) + 8d\lambda^3 - 2\lambda^2 b_2^2 + 8c\lambda^2(\sum_{i=1}^{n+1} b_i^2) + 16\lambda^2 c b_2^2,
\]
respectively. Since both of them are equal to zero, we get \(b_2 = 0\), which is a contradiction. So we can say that \(b_i = 0, \ i = 2, \cdots, k\). The equation (26) can be rewritten as
\[
\text{tr}(2A)(4\lambda^2 \sum_{i=2}^{k} x_i^2 + b_i^2 + \sum_{i=k+1}^{n+1} b_i^2)(\lambda \sum_{i=2}^{k} x_i^2 - d) = 0
\]
\[
-8\lambda^3(\sum_{i=2}^{k} x_i^2)(\lambda \sum_{i=2}^{k} x_i^2 - d) + c(4\lambda^2 \sum_{i=2}^{k} x_i^2 + b_i^2 + \sum_{i=k+1}^{n+1} b_i^2)^2 = 0. \tag{30}
\]
Then the coefficients of \((\sum_{i=2}^{k} x_i^2)^2, \ \sum_{i=2}^{k} x_i^2\) and the constant term of the left side of (30) are equal to
\[
4\lambda^3(\text{tr}(2A) - 2\lambda + 4c\lambda),
\]
\[
\text{tr}(2A)(-4d\lambda^2 + (b_1^2 + \sum_{i=k+1}^{n+1} b_i^2)\lambda) + 8d\lambda^3 + 8c(b_1^2 + \sum_{i=k+1}^{n+1} b_i^2)\lambda^2
\]
and

\[-\text{tr}(2A) d(b_1^2 + \sum_{i=k+1}^{n+1} b_i^2) + c(b_1^2 + \sum_{i=k+1}^{n+1} b_i^2)^2,\]

respectively. They must be equal to zero. Substituting $\text{tr}(2A) = 2(k-1)\lambda$ into the above coefficients, we have

\[k - 1 = 1 - 2c,\]

\[(k - 1)(-4d\lambda + b_1^2 + \sum_{i=k+1}^{n+1} b_i^2) + 4d\lambda + 4c(b_1^2 + \sum_{i=k+1}^{n+1} b_i^2) = 0\]

and

\[-2(k-1)d\lambda(b_1^2 + \sum_{i=k+1}^{n+1} b_i^2) + c(b_1^2 + \sum_{i=k+1}^{n+1} b_i^2)^2 = 0.\]

Substituting the first equation into the second one and the third one, we find

\[(b_1^2 + \sum_{i=k+1}^{n+1} b_i^2) + 8cd\lambda + 2c(b_1^2 + \sum_{i=k+1}^{n+1} b_i^2) = 0\]

and

\[-2d\lambda + 4cd\lambda + c(b_1^2 + \sum_{i=k+1}^{n+1} b_i^2) = 0.\]

Multiplying the number 2 at both sides of the second equation and subtracting it from the first equation, we get $4d\lambda = -(b_1^2 + \sum_{i=k+1}^{n+1} b_i^2)$. Substituting this into the first equation, we find $b_1^2 + \sum_{i=k+1}^{n+1} b_i^2 = 0$, which is a contradiction. So we may assume that $\lambda_i = 0$ implies that $b_i = 0$, $i = 1, \ldots, n+1$. Thus $(Ax + b, x) + d = 0$ can be written as $\sum_{i=1}^{k} \lambda_i x_i^2 + \sum_{i=1}^{k} b_i x_i + d = 0$ or $\sum_{i=1}^{k} \lambda_i(x_i + \frac{b_i}{2\lambda_i})^2 = e$ for a constant $e$ and the equation (26) can be given as

\[
\text{tr}(2A)\{4 \sum_{i=1}^{k} \lambda_i^2(x_i + \frac{b_i}{2\lambda_i})^2(\sum_{i=1}^{k} \lambda_i x_i^2 - d) - (8(\sum_{i=1}^{k} \lambda_i^3(x_i + \frac{b_i}{2\lambda_i})^2)(\sum_{i=1}^{k} \lambda_i x_i^2 - d)

+ c(4 \sum_{i=1}^{k} \lambda_i^2(x_i + \frac{b_i}{2\lambda_i})^2\} = 0
\]

or

\[
(\sum_{i=1}^{k} \lambda_i^2(\text{tr}(2A)-2\lambda_i)(x_i + \frac{b_i}{2\lambda_i})^2)(\sum_{i=1}^{k} \lambda_i x_i^2 - d)+4c(\sum_{i=1}^{k} \lambda_i^2(x_i + \frac{b_i}{2\lambda_i})^2)^2 = 0. \tag{31}
\]

Suppose $b_1 \neq 0$. Locally we may consider $M$ as the graph of the function $x_1 = \pm \sqrt{\frac{1}{\lambda_1}(e - \sum_{i=2}^{k} \lambda_i(x_i + \frac{b_i}{2\lambda_i})^2 - \frac{b_1}{2\lambda_1})}$. Substituting this function into (31)
we find
\[ g(x_2, \ldots, x_k)(e - d + \frac{b_i^2}{4\lambda_i} - \sum_{i=2}^{k} \frac{b_i^2}{4\lambda_i} - \sum_{i=2}^{k} b_i x_i) \pm b_i \sqrt{\frac{1}{\lambda_1} (e - \sum_{i=2}^{k} \lambda_i (x_i + \frac{b_i}{2\lambda_i})^2) + 4ch(x_2, \ldots, x_k)^2 = 0,} \]
where
\[ g(x_2, \ldots, x_k) = \sum_{i=2}^{k} \lambda_i^2 (\text{tr}(2A) - 2\lambda_i) (x_i + \frac{b_i}{2\lambda_i})^2 + \lambda_1 (\text{tr}(2A) - 2\lambda_1) (e - \sum_{i=2}^{k} \lambda_i (x_i + \frac{b_i}{2\lambda_i})^2) \]
and
\[ h(x_2, \ldots, x_k) = \sum_{i=2}^{k} \lambda_i^2 (x_i + \frac{b_i}{2\lambda_i})^2 + \lambda_1 (e - \sum_{i=2}^{k} \lambda_i (x_i + \frac{b_i}{2\lambda_i})^2). \]
If \( g(x_2, \ldots, x_k) \) is not identically zero, then a rational function is equal to a irrational function because of (32). So we have \( h(x_2, \ldots, x_k) = 0 \), which implies that \( \lambda_i = \lambda_1, \ i = 2, \ldots, k \) and \( e = 0 \). This implies that \( \sum_{i=1}^{k} \lambda_i (x_i + \frac{b_i}{2\lambda_i})^2 = e = 0 \) or \( \lambda_1 \sum_{i=1}^{k} (x_i + \frac{b_i}{2\lambda_i})^2 = 0 \). It is a contradiction. So we may conclude that \( b_i = 0, \ i = 1, \ldots, k \). Thus equation (26) can be written as
\[ -\text{tr}(2A)\langle 2Ax, 2Ax \rangle (2d) + \langle (2A)^2x, 2Ax \rangle (2d) + c\langle 2Ax, 2Ax \rangle^2 = 0 \]
or
\[ -\text{tr}(A)\langle Ax, Ax \rangle d + \langle A^2x, Ax \rangle d + c\langle Ax, Ax \rangle^2 = 0. \]
By this and similar arguments we have \( \lambda_i = \lambda_1, \ i = 2, \ldots, k \). This implies that if \( k = n + 1 \), then \( M \) is a hypersphere and if \( k < n + 1 \), then \( M \) is a spherical cylinder. Combining results in Case 1 and Case 2, we have the following proposition.

**Proposition 4.5.** If a quadric hypersurface \( M \) described by (16) in \( E^{n+1} \) satisfies \( \langle \Delta x, x \rangle = c \) for a constant \( c \), then it is one of the followings:

1. a minimal quadric hypersurface.
2. a nonminimal quadric hypersurface described by \( \langle Ax, x \rangle = 0 \) for a diagonal matrix \( A \).
3. a hypersphere.
4. a spherical cylinder.

**References**


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