STABILITY BY KRASNOSELSKII’S FIXED POINT THEOREM FOR NONLINEAR FRACTIONAL DYNAMIC EQUATIONS ON A TIME SCALE

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Abstract. In this paper, we give sufficient conditions to guarantee the asymptotic stability of the zero solution to a kind of nonlinear fractional dynamic equations of order $\alpha$ ($1 < \alpha < 2$). By using the Krasnosel’skii’s fixed point theorem in a weighted Banach space, we establish new results on the asymptotic stability of the zero solution provided $f(t,0) = 0$, which include and improve some related results in the literature.

1. Introduction

The concept of time scales analysis is a fairly new idea. In 1988, it was introduced by the German mathematician Stefan Hilger in his Ph.D. thesis [21]. It combines the traditional areas of continuous and discrete analysis into one theory. After the publication of two textbooks in this area by Bohner and Peterson [11] and [12], more and more researchers were getting involved in this fast-growing field of mathematics. The study of dynamic equations brings together the traditional research areas of differential and difference equations. It allows one to handle these two research areas at the same time, hence shedding light on the reasons for their seeming discrepancies. In fact, many new results for the continuous and discrete cases have been obtained by studying the more general time scales case (see [2], [3], [4], [7]-[10], [22], [28] and the references therein).
Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1], [3], [5], [13], [14], [19], [20], [23], [24], [26], [28] and the references therein.

There is no doubt that the Lyapunov method have been used successfully to investigate stability properties of wide variety of ordinary, functional and partial equations. Nevertheless, the application of this method to problem of stability in differential equations with delay has encountered serious difficulties if the delay is unbounded or if the equation has unbounded term. It has been noticed that some of these difficulties vanish by using the fixed point technic. Other advantages of fixed point theory over Lyapunov’s method is that the conditions of the former are average while those of the latter are pointwise (see [6], [15]-[18], [25] and references therein).

Recently, Ge and Kou [19] investigated the asymptotic stability of the zero solution of the following nonlinear fractional differential equation

\[
\begin{align*}
C D^\alpha_{0+} x(t) &= f(t, x(t)), \quad t \geq 0, \\
 x(0) &= x_0, \quad x'(0) = x_1,
\end{align*}
\]

where \( C D^\alpha_{0+} \) is the standard Caputo’s fractional derivative of order \( 1 < \alpha < 2 \). By employing the Krasnoselskii’s fixed point theorem in a weighted Banach space, the authors obtained stability results.

In [3], Ahmadkhanlu and Jahanshahi studied the initial value problem for the fractional dynamic equations on time scales

\[
\begin{align*}
\frac{C^\alpha T}{C T} D^\alpha x(t) &= f(t, x(t)), \quad t \in [t_0, t_0 + a]_T, \\
x(0) &= x_0,
\end{align*}
\]

where \( C T D^\alpha \) is the Caputo’s fractional derivative of order \( 0 < \alpha < 1 \). By using the Banach and Schauder fixed point theorems, the existence and uniqueness of solutions has been established.

Motivated by all the works above, in this paper we concentrate on the asymptotic stability of the zero solution for the nonlinear fractional
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dynamic equation

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\frac{C}{T}D_{0+}^{\alpha} x(t) = f(t, x(t)), \quad t \in [0, \infty)_T, \\
x(0) = x_0, \quad x^\triangle(0) = x_1,
\end{array} \right.
\end{aligned}
\tag{1}
\]

where \( T \) is an unbounded above time scale with \( 0 \in T \), \( \frac{C}{T}D_{0+}^{\alpha} \) is the Caputo’s fractional derivative on \( T \) of order \( 1 < \alpha < 2 \), \( x_0, x_1 \in \mathbb{R} \), \( f : [0, \infty)_T \times \mathbb{R} \to \mathbb{R} \) is a rd-continuous function with \( f(t, 0) \equiv 0 \). To show the asymptotic stability of the zero solution, we transform (1) into an integral equation and then use Krasnoselskii’s fixed point theorem. The obtained integral equation is the sum of two mappings, one is a contraction and the other is compact.

This paper is organized as follows. In section 2, we introduce some notations and lemmas, and state some preliminaries results needed in later sections. Also, we present the inversion of (1) and the Krasnoselskii’s fixed point theorem. For details on Krasnoselskii’s theorem we refer the reader to [27]. In Section 3, we give and prove our main results on stability. The results obtained here extend the work of Ahmadkhanlu and Jahanshahi [19].

2. Preliminaries

In this section, some notations, definitions and lemmas which will be used in next section are recalled and introduced. At first, we use \( C_{rd}([0, \infty)_T) \) for a Banach space of rd-continuous functions with the norm

\[
\|x\|_\infty = \sup \{ |x(t)|, \; t \in [0, \infty)_T \},
\]

where \([0, \infty)_T\) is an interval.

A time scale \( T \) is an arbitrary nonempty closed subset of the real numbers. The calculus on time scales was initiated by Hilger [21] in order to create a theory that can unify and extend discrete and continuous analysis. In the definition of the derivative an important role is played by the so-called forward and backward jump operators [11].

**Definition 2.1** ([11]). For \( t \in T \), define the forward jump operator \( \sigma : T \to T \)

\[
\sigma(t) = \inf \{ s \in T : s > t \},
\]

while the backward jump operator \( \rho : T \to T \) is defined by

\[
\rho(t) = \sup \{ s \in T : s < t \}.
\]
In this definition, in addition we put \( \sigma(\max T) = \max T \), if there exists a finite \( \max T \), and \( \rho(\min T) = \min T \), if there exists a finite \( \min T \). Obviously both \( \sigma(t) \) and \( \rho(t) \) are in \( T \) when \( t \in T \). This is because of our assumption that \( T \) is a closed subset of \( \mathbb{R} \). Let \( t \in T \).

If \( \sigma(t) > t \), we say that \( t \) is right-scattered, while if \( \rho(t) < t \) we say that \( t \) is left-scattered. Also, if \( t < \max T \) and \( \sigma(t) = t \), then \( t \) is called right-dense, and if \( t > \min T \) and \( \rho(t) = t \), then \( t \) is called left-dense.

Points that are right-scattered and left-scattered at the same time are called isolated.

**Definition 2.2** ([11]). A function \( f : T \to \mathbb{R} \) is right-dense continuous (or \( rd \)-continuous) provided that it is continuous at all right-dense points of \( T \) and its left-sided limits exist (finite) at left-dense points of \( T \). The set of all right-dense continuous functions on \( T \) is denoted by \( C_{rd}(T) \). Similarly, a function \( f : T \to \mathbb{R} \) is left-dense continuous provided that it is continuous at all left-dense points of \( T \) and its right-sided limits exist (finite) at right-dense points of \( T \). The set of all left-dense continuous functions on \( T \) is denoted by \( C_{ld}(T) \).

**Definition 2.3** (Delta derivative [11]). Let \( f : T \to \mathbb{R} \) be a function and \( t \in T \). Then the delta derivative (or \( \Delta \)-derivative) of \( f \) at the point \( t \) is defined to be the number \( f^\Delta(t) \) (provided it exists) with the property that for each \( \varepsilon > 0 \) there is a neighborhood \( U \) of \( t \) in \( T \) such that

\[
|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \text{ for all } s \in U.
\]

**Definition 2.4** (Delta integral [11]). Let \( [a,b]_T \) be a closed bounded interval in \( T \). A function \( F : [a,b]_T \to \mathbb{R} \) is called a delta antiderivative of a function \( F : [a,b]_T \to \mathbb{R} \) provided that \( F \) is \( rd \)-continuous on \( [a,b]_T \) and delta differentiable on \( [a,b)_T \) and \( F^\Delta(t) = f(t) \) for all \( t \in [a,b)_T \). Then we define the \( \Delta \)-integral from \( a \) to \( b \) of \( f \) by

\[
\int_a^b f(t) \Delta t = F(b) - F(a).
\]

All \( rd \)-continuous bounded functions on \( [a,b]_T \) are delta integrable from \( a \) to \( b \). For a more general treatment of the delta integral on time scales (Riemann and Lebesgue integration on time scales), see [4].

**Definition 2.5** ([3]). Suppose \( T \) is a time scale, \( [a,b]_T \subseteq T \) and the function \( x \) is an integrable function on \( [a,b]_T \), then \( \Delta \)-fractional integral of \( x \) is defined by the following relation

\[
\Delta^\alpha \int_{a+}^{b} x(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t - s)^{\alpha - 1} x(s) \Delta s,
\]
where $\Gamma (\alpha )$ is the Euler Gamma function.

**Definition 2.6** ([3]). Let $x : \mathbb{T} \rightarrow \mathbb{R}$ be a function. The Caputo $\Delta$-fractional derivative of $x$ is defined by

$$
\frac{C_{\Delta} D_{a+}^{\alpha} x(t)}{t} = \frac{1}{\Gamma (n-\alpha )} \int_{a}^{t} (t-s)^{n-\alpha -1} x^{\Delta^n} (s) \Delta s.
$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer of $\alpha$.

**Lemma 2.7** ([3]). Let $0 < \Re (\alpha ) < 1$. Suppose $x \in C_{rd} ([0, \infty )_{\mathbb{T}})$ and $x^{\Delta}$ exists almost everywhere on any bounded interval of $[0, \infty )_{\mathbb{T}}$. Then

$$
\frac{\Delta T I_{0+}^{\alpha} C_{\Delta} D_{a+}^{\alpha} x(t)}{t} = x(t) - x(0).
$$

**Remark 2.8.** From Definitions 2.5, 2.6 and Lemma 2.7, it is easy to see that

1. Let $0 < \Re (\alpha ) < 1$. If $x$ is rd-continuous on $[0, \infty )_{\mathbb{T}}$, then $\frac{C_{\Delta} D_{a+}^{\alpha} x(t)}{t} = x(t)$ holds for all $t \in [0, \infty )_{\mathbb{T}}$.

2. The Caputo derivative of a constant is equal to zero.

The following Banach space plays a fundamental role in our discussion. Let $h : [0, \infty )_{\mathbb{T}} \rightarrow [1, +\infty )$ be a strictly increasing rd-continuous function with $h(0) = 1$, $h(t) \rightarrow \infty$ as $t \rightarrow \infty$, $h(s) h(t-s) \leq h(t)$ for all $0 \leq s \leq t < \infty$. Let

$$
E = \left\{ x \in C_{rd} ([0, \infty )_{\mathbb{T}}) : \sup_{t \in (0, \infty )_{\mathbb{T}}} |x(t)| / h(t) < \infty \right\}.
$$

Then $E$ is a Banach space equipped with the norm $\|x\| = \sup_{t \in (0, \infty )_{\mathbb{T}}} |x(t)| / h(t)$. For more properties of this Banach space, see [24]. Moreover, let

$$
\|\varphi\| = \max \{|\varphi (s)| : 0 \leq s \leq t\},
$$

for any $t \in [0, \infty )_{\mathbb{T}}$, any given $\varphi \in C_{rd} ([0, \infty )_{\mathbb{T}})$ and let $\Im (\varepsilon ) = \{ x \in E : \|x\| \leq \varepsilon \}$ for any $\varepsilon > 0$.

**Lemma 2.9.** Let $r \in C_{rd} ([0, \infty )_{\mathbb{T}})$. Then $x \in C_{rd} ([0, \infty )_{\mathbb{T}})$ is a solution of the Cauchy type problem

$$
\frac{C_{\Delta} D_{a+}^{\alpha} x(t)}{t} = r(t), \; t \in [0, \infty )_{\mathbb{T}}, \; 1 < \alpha < 2,
$$

if and only if $x$ is a solution of the Cauchy type problem

$$
\begin{align*}
x^{\Delta} (t) & = \frac{\Delta}{\Gamma (\alpha -1)} I_{0+}^{\alpha -1} r(t) + x_1, \\
x(0) & = x_0.
\end{align*}
$$
Proof. Clearly, if $\psi \in C_{rd}([0, \infty)_T)$, then $\Delta T I_{0+}^\gamma \psi(0) = 0$ with $0 < \gamma < 1$.

1) Let $x \in C_{rd}([0, \infty)_T)$ be a solution of (2). For any $t \in [0, \infty)_T$, Definition 2.6 shows that

$$\mathcal{CD}_{0+}^\alpha x(t) = \mathcal{CD}_{0+}^{\alpha-1} x^\Delta(t) = r(t).$$

According to Lemma 2.7, we have

$$x^\Delta(t) = x^\Delta(0) + \Delta T I_{0+}^{\alpha-1} r(t) = x^\Delta(0) + x_1,$$

which means that $x$ is a solution of (3).

2) Let $x$ be a solution of (3). For any $t \in [0, \infty)_T$, by Remark 2.8, it is easy to see that

$$\mathcal{CD}_{0+}^\alpha x(t) = \mathcal{CD}_{0+}^{\alpha-1} x^\Delta(t) = \mathcal{CD}_{0+}^{\alpha-1} \Delta T I_{0+}^{\alpha-1} r(t) + \mathcal{CD}_{0+}^{\alpha-1} x_1 = r(t).$$

Besides, note that $r \in C_{rd}([0, \infty)_T)$, we have

$$x^\Delta(0) = \Delta T I_{0+}^{\alpha-1} r(0) + x_1 = x_1.$$

Lemma 2.10. Let $k \in (0, \infty)$. Then $x \in C_{rd}([0, \infty)_T)$ is a solution of (1) if and only if

$$x(t) = x_0 e_{\ominus k}(t, 0) + \frac{1 - e_{\ominus k}(t, 0)}{k} x_1 + k \int_0^t e_{\ominus k}(t, s) x(s) \Delta s + \frac{1}{\Gamma(\alpha - 1)} \int_0^t \int_u^t e_{\ominus k}(t, s) (s - u)^{\alpha-2} f(s, x(s)) \Delta u,$$

(4)

Proof. Let $x \in C_{rd}([0, \infty)_T)$ be a solution of (1). From Lemma 2.9, we have

$$\begin{cases} x^\Delta(t) = \Delta T I_{0+}^{\alpha} (f(t, x(t))) + x_1, & t \in [0, \infty)_T, \\ x(0) = x_0, & t \in [0, \infty)_T. \end{cases}$$

Then

$$\begin{cases} x^\Delta(t) = \frac{1}{\Gamma(\alpha - 1)} \int_0^t (t - s)^{\alpha-2} f(s, x(s)) ds + x_1, & t \in [0, \infty)_T, \\ x(0) = x_0, & t \in [0, \infty)_T. \end{cases}$$

Rewrite (5) as

$$\begin{cases} x^\Delta(t) + k x(t) = k x(t) + \frac{1}{\Gamma(\alpha - 1)} \int_0^t (t - s)^{\alpha-2} f(s, x(s)) ds + x_1, & t \in [0, \infty)_T, \\ x(0) = x_0, & t \in [0, \infty)_T. \end{cases}$$

By the variation of constants formula, we obtain (4). Since each step is reversible, the converse follows easily. This completes the proof.

\[\square\]
Definition 2.11. The trivial solution $x = 0$ of (1) is said to be
(i) stable in Banach space $E$, if for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $|x_0| + |x_1| \leq \delta$ implies that the solution $x(t) = x(t, x_0, x_1)$ exists for all $t \in [0, \infty)$ and satisfies $\|x\| \leq \varepsilon$.

(ii) asymptotically stable, if it is stable in Banach space $E$ and there exists a number $\sigma > 0$ such that $|x_0| + |x_1| \leq \sigma$ implies $\lim_{t \to \infty} \|x(t)\| = 0$.

Lastly in this section, we state Krasnoselskii’s fixed point theorem which enables us to prove the asymptotic stability of the zero solution to (1). For its proof we refer the reader to [27].

Theorem 2.12 (Krasnoselskii [27]). Let $\Omega$ be a non-empty closed convex subset of a Banach space $(S, \|\|)$. Suppose that $A$ and $B$ map $\Omega$ into $S$ such that
(i) $Ax + By \in \Omega$ for all $x, y \in \Omega$,
(ii) $A$ is continuous and $A\Omega$ is contained in a compact set of $S$,
(iii) $B$ is a contraction with constant $l < 1$.
Then there is a $x \in \Omega$ with $Ax + Bx = x$.

In order to prove (ii), the following modified compactness criterion is needed.

Theorem 2.13 ([24]). Let $\mathcal{M}$ be a subset of the Banach space $E$. Then $\mathcal{M}$ is relatively compact in $E$ if the following conditions are satisfied
(i) $\{x(t)/h(t) : x \in \mathcal{M}\}$ is uniformly bounded,
(ii) $\{x(t)/h(t) : x \in \mathcal{M}\}$ is equicontinuous on any compact interval of $\mathbb{R}^+$,
(iii) $\{x(t)/h(t) : x \in \mathcal{M}\}$ is equiconvergent at infinity i.e. for any given $\varepsilon > 0$, there exists a $T_0 > 0$ such that for all $x \in \mathcal{M}$ and $t_1, t_2 > T_0$, if holds
$$|x(t_2)/h(t_2) - x(t_1)/h(t_1)| < \varepsilon.$$

3. Main results

Before stating and proving the main results, we introduce the following hypotheses.

(h1) There exists a constant $\beta_1 \in (0, 1)$ such that
$$e_{\ominus k}(t, 0)/h(t) \in BC([0, \infty)_\tau) \cap L^1\Delta([0, \infty)_\tau),$$
(6) $$k \int_0^t e_{\ominus k}(t, 0)/h(u) \Delta u \leq \beta_1 < 1.$$
There exists constants $\eta > 0$, $\beta_2 \in (0, 1 - \beta_1)$ and a continuous function $\tilde{f}: [0, \infty)_T \times (0, \eta] \rightarrow \mathbb{R}^+$ such that

$$
\left| \frac{f(t, \nu h(t))}{h(t)} \right| \leq \tilde{f}(t, |\nu|),
$$

holds for all $t \in [0, \infty)_T$, $0 < |\nu| \leq \eta$ and

$$
\sup_{t \in [0, \infty)_T} \int_{0}^{t} K(t-u) \frac{\tilde{f}(u,r)}{h(t-u)} \frac{\Delta u}{r} \leq \beta_2 < 1 - \beta_1,
$$

holds for every $0 < r \leq \eta$, where $\tilde{f}(t,r) \in L^1([0, \infty)_T)$ in $t$ for fixed $r$, and

$$
K(t-u) = \begin{cases} 
\frac{1}{\Gamma(\alpha-1)} \int_{0}^{t} e_{\alpha-k}(t,s)(s-u)^{\alpha-2} \Delta s, & t-u \geq 0, \\
0, & t-u < 0.
\end{cases}
$$

**Theorem 3.1.** Suppose that (h1) and (h2) hold. Then the trivial solution $x = 0$ of (1) is stable in Banach space $E$.

*Proof.* For any given $\varepsilon > 0$, we first prove the existence of $\delta > 0$ such that

$$
|x_0| + |x_1| < \delta \text{ implies } \|x\| \leq \varepsilon.
$$

In fact, according to (6), there exists a constant $M_1 > 0$ such that

$$
\frac{e_{\alpha-k}(t,0)}{h(t)} \leq M_1.
$$

Let $0 < \delta \leq \frac{(1-\beta_1-\beta_2)k}{M_1 k + 1 + M_1 \varepsilon}$. Consider the non-empty closed convex subset $\Im(\varepsilon) \subseteq E$, for $t \in [0, \infty)_T$, we denote two mapping $A$ and $B$ on $\Im(\varepsilon)$ as follows

$$
Ax(t) = \frac{1}{\Gamma(\alpha-1)} \int_{0}^{t} \int_{u}^{t} e_{\alpha-k}(t,s)(s-u)^{\alpha-2} \Delta s f(u,x(u)) \Delta u
$$

and

$$
Bx(t) = e_{\alpha-k}(t,0) x_0 + \frac{1 - e_{\alpha-k}(t,0)}{k} x_1 + k \int_{0}^{t} e_{\alpha-k}(t,s) x(s) \Delta s.
$$

Obviously, for $x \in \Im(\varepsilon)$, both $Ax$ and $Bx$ are continuous functions on $[0, \infty)_T$. Furthermore, for $x \in \Im(\varepsilon)$, by (6)-(8) for any $t \in [0, \infty)_T$, we
have

\[
\frac{|Ax(t)|}{h(t)} \leq \int_0^t K(t-u) \frac{|f(u,x(u))|}{h(u)} \Delta u
\]
\[
\leq \int_0^t \frac{K(t-u)}{h(t-u)} \tilde{f} \left( u, \frac{|x(u)|}{h(u)} \right) \Delta u
\]
\[
\leq \beta_2 \|x\| \leq \beta_2 \varepsilon < \infty,
\]
(13)

and

\[
\frac{|Bx(t)|}{h(t)} = \left| \frac{e_{\odot k}(t,0)}{h(t)} x_0 + \frac{1 - e_{\odot k}(t,0)}{kh(t)} x_1 + k \int_0^t \frac{e_{\odot k}(t,s)}{h(t)} x(s) \Delta s \right|
\]
\[
\leq M_1 |x_0| + \frac{1 + M_1}{k} |x_1| + k \int_0^\infty \frac{e_{\odot k}(u,0)}{h(u)} \Delta u \|x\|
\]
\[
\leq M_1 |x_0| + \frac{1 + M_1}{k} |x_1| + \beta_1 \varepsilon < \infty.
\]
(14)

Then \( A \mathcal{I}(\varepsilon) \subseteq E \) and \( B \mathcal{I}(\varepsilon) \subseteq E \). Next, we shall use Theorem 2.12 to prove there exists at least one fixed point of the operator \( A + B \) in \( \mathcal{I}(\varepsilon) \). Here, we divide the proof into three steps.

**Step 1.** We prove that \( Ax + By \in \mathcal{I}(\varepsilon) \) for all \( x, y \in \mathcal{I}(\varepsilon) \).

For any \( x, y \in \mathcal{I}(\varepsilon) \), from (13) and (14), we obtain that

\[
\sup_{t \in [0,\infty)} \frac{|Ax(t) + By(t)|}{h(t)}
\]
\[
= \sup_{t \in [0,\infty)} \left\{ \left| \frac{e_{\odot k}(t,0)}{h(t)} x_0 + \frac{1 - e_{\odot k}(t,0)}{kh(t)} x_1 + k \int_0^t \frac{e_{\odot k}(t,s)}{h(t)} x(s) \Delta s \right| + \int_0^t K(t-u) \tilde{f}(u,x(u)) \Delta u \right\}
\]
\[
\leq M_1 |x_0| + \frac{1 + M_1}{k} |x_1| + k \int_0^\infty \frac{e_{\odot k}(u,0)}{h(u)} \Delta u \|y\| + \beta_2 \|x\|
\]
\[
\leq \frac{M_1 k + 1 + M_1}{k} \delta + \beta_1 \varepsilon + \beta_2 \varepsilon \leq \varepsilon,
\]

which implies \( Ax + By \in \mathcal{I}(\varepsilon) \) for all \( x, y \in \mathcal{I}(\varepsilon) \).

**Step 2.** It is easy to see that \( A \) is continuous. Now we only prove that \( A \mathcal{I}(\varepsilon) \) is a relatively compact in \( E \). In fact, from (13), we get that \( \{x(t)/h(t) : x \in \mathcal{I}(\varepsilon) \} \) is uniformly bounded in \( E \). Moreover, a classical theorem states the fact that the convolution of an \( L^1_{\Delta} \)-function with a function tending to zero, does also tend to zero. Then we conclude that
for \( t - u \in [0, \infty) \), we have

\[
0 \leq \lim_{t \to \infty} \frac{K(t-u)}{h(t-u)} \leq \lim_{t \to \infty} \frac{1}{\Gamma(\alpha-1)} \int_{0}^{t} e_{\alpha}^{k} \frac{(s-u)^{\alpha-2}}{h(t-u) \cdot h(s-u)} \Delta s
\]

\[
= \lim_{t \to \infty} \frac{1}{\Gamma(\alpha-1)} \int_{0}^{t} e_{\alpha}^{k} \frac{(t-u,s)^{\alpha-2}}{h(t-u-s) \cdot h(s)} \Delta s = 0,
\]

due to the fact \( \lim_{t \to \infty} \frac{(t-u)^{\alpha-2}}{h(t)} = 0 \). Together with the continuity of \( K \) and \( h \), we get that there exists a constant \( M_2 > 0 \) such that

\[
\left| \frac{K(t-u)}{h(t-u)} \right| \leq M_2,
\]

and for any \( T_0 \in [0, \infty) \), the function \( K(t-u)h(u)/h(t) \) is uniformly continuous on \( \{(t,u) : 0 \leq u \leq t \leq T_0 \} \). For any \( t_1, t_2 \in [0, T_0] \cap \mathbb{T} \), \( t_1 < t_2 \), we have

\[
\left| \frac{Ax(t_2)}{h(t_2)} - \frac{Ax(t_1)}{h(t_1)} \right| = \left| \int_{0}^{t_2} \frac{K(t_2-u)}{h(t_2)} f(u,x(u)) \Delta u - \int_{0}^{t_1} \frac{K(t_1-u)}{h(t_1)} f(u,x(u)) \Delta u \right|
\]

\[
\leq \int_{0}^{t_1} \left| \frac{K(t_2-u)}{h(t_2)} - \frac{K(t_1-u)}{h(t_1)} \right| \left| f(u,x(u)) \right| \Delta u + \int_{t_1}^{t_2} \frac{K(t_2-u)}{h(t_2)} f(u,\varepsilon) \Delta u
\]

\[
\leq \int_{0}^{t_1} \left| \frac{K(t_2-u)h(u)}{h(t_2)} - \frac{K(t_1-u)h(u)}{h(t_1)} \right| \frac{h(t)}{h(u)} f(u,\varepsilon) \Delta u + M_2 \int_{t_1}^{t_2} f(u,\varepsilon) \Delta u \to 0,
\]

as \( t_2 \to t_1 \), which means that \( \{x(t)/h(t) : x \in \mathbb{Z}(\varepsilon)\} \) is equicontinuous on any compact interval of \([0, \infty) \). By Theorem 2.13, in order to show that \( A\mathbb{Z}(\varepsilon) \) is a relatively compact set of \( E \), we only need to prove that \( \{x(t)/h(t) : x \in \mathbb{Z}(\varepsilon)\} \) is equiconvergent at infinity. In fact, for any \( \varepsilon_1 > 0 \), there exists a \( L > 0 \) such that

\[
M_2 \int_{L}^{\infty} \frac{h(t)}{h(t)} f(u,\varepsilon) \Delta u \leq \frac{\varepsilon_1}{3}.
\]

According to (15), we get that

\[
\lim_{t \to \infty} \sup_{u \in [0,L] \cap \mathbb{T}} \frac{K(t-u)}{h(t-u)} \leq \max \left\{ \lim_{t \to \infty} \frac{K(t-L)}{h(t-L)} , \lim_{t \to \infty} \frac{K(t)}{h(t)} \right\} = 0.
\]
Thus, there exists $T > L$ such that $t_1, t_2 \geq T$, we have

$$
\sup_{u \in [0, L] \cap T} \left| \frac{K(t_2 - u)h(u)}{h(t_2)} - \frac{K(t_1 - u)h(u)}{h(t_1)} \right|
\leq \sup_{u \in [0, L] \cap T} \left| \frac{K(t_2 - u)}{h(t_2)} \right| + \sup_{u \in [0, L] \cap T} \left| \frac{K(t_1 - u)}{h(t_1 - u)} \right|
\leq \frac{\varepsilon_1}{3} \left( \int_0^\infty \tilde{f}(u, \varepsilon) \Delta u \right)^{-1}.
$$

Therefore, for $t_1, t_2 \geq T$,

$$
\left| \frac{Ax(t_2)}{h(t_2)} - \frac{Ax(t_1)}{h(t_1)} \right|
= \left| \int_0^{t_2} \frac{K(t_2 - u)}{h(t_2)} f(u, x(u)) \Delta u - \int_0^{t_1} \frac{K(t_1 - u)}{h(t_1)} f(u, x(u)) \Delta u \right|
\leq \int_0^L \left| \frac{K(t_2 - u)h(u)}{h(t_2)} - \frac{K(t_1 - u)h(u)}{h(t_1)} \right| \tilde{f}(u, \varepsilon) \Delta u
+ \int_L^{t_2} \frac{K(t_2 - u)}{h(t_2 - u)} \tilde{f}(u, \varepsilon) \Delta u + \int_L^{t_1} \frac{K(t_1 - u)}{h(t_1 - u)} \tilde{f}(u, \varepsilon) \Delta u
\leq \frac{\varepsilon_1}{3} + 2M_2 \int_L^\infty \tilde{f}(u, \varepsilon) \Delta u \leq \varepsilon_1.
$$

Hence the required conclusion is true.

**Step 3.** we claim that $B : \mathcal{S}(\varepsilon) \to E$ is a contraction mapping.

In fact, for any $x, y \in \mathcal{S}(\varepsilon)$, from (6), we obtain that

$$
\sup_{t \in [0, \infty) \cap T} \left| \frac{Bx(t)}{h(t)} - \frac{By(t)}{h(t)} \right|
= \sup_{t \in [0, \infty) \cap T} \left\{ \frac{k}{h(t)} \int_0^t e_{\ominus k} (t, u) x(u) \Delta u - \frac{k}{h(t)} \int_0^t e_{\ominus k} (t, u) y(u) \Delta u \right\}
\leq \sup_{t \in [0, \infty) \cap T} k \int_0^t e_{\ominus k} (t, u) \frac{|x(u) - y(u)|}{h(t - u)} \Delta u
\leq k \int_0^t e_{\ominus k} (t, u) \frac{|x(u) - y(u)|}{h(t - u)} \Delta u
\leq \beta_1 \|x - y\|.
$$

By Theorem 2.12, we know that there exists at least one fixed point of the operator $A+B$ in $\mathcal{S}(\varepsilon)$. Finally, for any $\varepsilon_2 > 0$, if $0 < \delta_1 \leq \frac{(1-\beta_1-\beta_2)k}{k\lambda_1+1+\lambda_1^2}\varepsilon_2$,
then $|x_0| + |x_1| \leq \delta_1$ implies that

$$
\|x\| = \sup_{t \in [0, \infty)^T} \left\{ \frac{e^{\otimes k}(t,0)}{h(t)} x_0 + \frac{1 - e^{\otimes k}(t,0)}{kh(t)} x_1 + k \int_0^t \frac{e^{\otimes k}(t,s)}{h(t)} x(s) \Delta u + \int_0^t \frac{K(t-u)}{h(t)} f(u,x(u)) \Delta u \right\}
$$

$$
\leq \sup_{t \in [0, \infty)^T} \left\{ \frac{e^{\otimes k}(t,0)}{h(t)} |x_0| + \frac{|1 - e^{\otimes k}(t,0)|}{kh(t)} |x_1|ight.

$$

$$
+ k \int_0^t \frac{e^{\otimes k}(t,u)}{h(t-u)h(u)} \Delta u + \int_0^t \frac{K(t-u)}{h(t-u)} |f(u,x(u))| \Delta u \right\}
\leq M_1 \delta_1 + \frac{1 + M_1}{k} \delta_1 + \beta_1 \|x\| + \beta_2 \|x\|
$$

$$
\leq \frac{kM_1 + 1 + M_1}{(1 - \beta_1 - \beta_2) k} \delta_1 \leq \varepsilon_2.
$$

Thus, we know that trivial solution of (1) is stable in Banach space $E$.

\[ \text{Theorem 3.2. Suppose that all conditions of Theorem 3.1 are satisfied,} \]

\[ (17) \lim_{t \to \infty} \frac{e^{\otimes k}(t,0)}{h(t)} = 0, \]

and for any $r > 0$, there exists a function $\varphi_r \in L^1_{\Delta}([0, \infty)^T)$, $\varphi_r(t) > 0$ such that $|u| \leq r$ implies

\[ (18) \left| f(t,u) \right| / h(t) \leq \varphi_r(t), \ a.e. \ t \in [0, \infty)^T. \]

Then the trivial solution of (1) is asymptotically stable.

\[ \text{Proof. First, it follows from Theorem 3.1 that the trivial solution of} \]

\[ (1) \text{is stable in the Banach space } E. \text{ Next, we shall show that the trivial solution } x = 0 \text{ of (1) is attractive. For any } r > 0, \text{ defining} \]

\[ \mathcal{S}_r = \left\{ x \in \mathcal{S} : \lim_{t \to \infty} x(t)/h(t) = 0 \right\}. \]

We only need to prove that $Ax + By \in \mathcal{S}_r$ for any $x, y \in \mathcal{S}_r$, i.e.

\[ \frac{Ax(t) + By(t)}{h(t)} \to 0 \text{ as } t \to \infty, \]
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where

\[\begin{align*}
Ax(t) + By(t) &= e_{\ominus k}(t, 0)x_0 + \frac{1 - e_{\ominus k}(t, 0)}{k}x_1 + k \int_0^t e_{\ominus k}(t, s)y(u)\Delta u \\
&+ \int_0^t K(t - u)f(u, x(u))\Delta u.
\end{align*}\]

In fact, for \(x, y \in \mathcal{H}^\alpha(r)\), based on the fact that used in the proof of Theorem 3.1 (Step 2), it follows from (6) and (17) that

\[\int_0^t e_{\ominus k}(t, u)y(u)\frac{h(t - u)}{h(u)}\Delta u \to 0,\]

and

\[\frac{K(t - u)}{h(t - u)} = \int_0^t e_{\ominus k}(t, u)\frac{(s - u)^{\alpha - 2}\Delta s}{\Gamma(\alpha - 1)} \to 0,\]

as \(t \to \infty\). Together with the hypothesis \(\varphi_r(t) \in L_{\alpha}^1([0, \infty)_T)\), we obtain that

\[\int_0^t \frac{K(t - u)}{h(t - u)}\frac{\varphi_r(u)}{h(u)}\Delta u \to 0,\]

as \(t \to \infty\). Thus we get the conclusion. \(\square\)

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References


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