FRACTIONAL DYNAMICAL SYSTEMS FOR VARIATIONAL INCLUSIONS INVOLVING DIFFERENCE OF OPERATORS

Awaïs Gul Khan*, Muhammad Aslam Noor, and Khalida Inayat Noor

Abstract. In the present paper, we propose some new fractional dynamical systems. These dynamical systems are associated with the variational inclusions involving difference of operators problem. The equivalence between the variational inclusion problems and the fixed point problems and as well as the resolvent equations are used to suggest fractional resolvent dynamical systems and fractional resolvent equation dynamical systems, respectively. We show that these dynamical systems converge $\alpha$-exponentially to the unique solution of variational inclusion problems under fewer restrictions imposed on operators and parameters. Several special cases also discussed.

1. Introduction

Variational inequality theory provides us a simple, natural, general and unified framework for studying a wide class of unrelated problems arising in elasticity, structural analysis, economics, optimization, oceanography, regional, physical and engineering sciences, etc., see [1-36] and the references therein. It is well known that the variational inequalities are equivalent to the fixed-point problems. This alternative formulation has played an important and fundamental part in developing a wide class of projection type methods for solving variational inequalities and complementarity problems. This equivalence has been used to analyzed the projected dynamical systems, in which the right-hand

---

Received September 6, 2018. Accepted January 14, 2019.
2010 Mathematics Subject Classification. Primary 49J40; Secondary 46T99, 47H05.

Key words and phrases. Fractional dynamical systems, Variational inclusions, Difference of operators
*Corresponding author
side of the ordinary differential equation is a projection operator. The novel feature of the projected dynamical systems is that the set of the stationary points of the dynamical system correspond to the set of the solution of the variational inequalities. Consequently, equilibrium problems which can be formulated in the setting of variational inequalities can now be studied in the framework of the dynamical systems. Xia and Wang [33] have shown that the projected dynamical systems can be used effectively in designing neural network for solving variational inequalities and related optimization problems.

Variational inequalities have been extended and generalized in different directions using novel and innovative techniques both for its own sake and for its applications. A useful and an important generalization of variational inequalities is a variational inclusion. It is well known the projection method and its variant forms including the Wiener-Hopf equations cannot be used to suggest projected type dynamical systems for solving the variational inclusion problems. These facts motivated us to use the technique of the resolvent operator. Using the resolvent operator technique, one can show that the variational inclusion problems are equivalent to the fixed point problems. It is interesting to note that the resolvent equations generalize and extend the Wiener-Hopf equations introduced by Robinson [29] and Shi [30], independently.

Motivated and inspired by the recent research work going on in the field of fractional calculus, in this paper, we use this alternative equivalent formulation to propose and analyze some new fractional dynamical systems related to the variational inclusion problem. It has been shown in [21] that the variational inclusions are equivalent to the resolvent equations. We use the resolvent equations technique to suggest an other fractional resolvent dynamical system associated with the variational inclusions. We show that these fractional dynamical systems converge $\alpha$-exponentially to the unique solution of variational inclusions involving difference of two monotone operators under some reasonable conditions. Some special cases are also discussed.

2. Formulation and Basic Results

Let $H$ be a real Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively.
For given monotone operators $T, A : H \to H$, consider a problem of finding $u \in H$ such that

\[(1)\quad 0 \in A(u) - Tu.\]

The problem of type (1) is called variational inclusion involving difference of monotone operators. The problem (1) considered by Moudafi [13] and Noor et al. [21, 22] recently using essentially two different techniques. Note that the difference of two monotone operators is not a monotone operator as contrast to the sum of two monotone operators. Due to this fact, the problem of finding a zero of the difference of two monotone operators is very difficult as compared to finding the zeros of the sum of monotone operators, see [1, 13, 21, 22].

We now discuss some special cases of problem (1).

**I.** If $A(\cdot) \equiv \partial \varphi(\cdot)$, the subdifferential of a proper, convex and lower-semicontinuous function $\varphi : H \to \mathbb{R} \cup \{\infty\}$, then problem (1) is equivalent to finding $u \in H$ such that

\[(2)\quad 0 \in \partial \varphi(u) - Tu,\]

or equivalently, find $u \in H$ such that

\[(3)\quad \langle Tu, v - u \rangle + \varphi(u) - \varphi(v) \leq 0, \quad \forall v \in H,
\]

which is known as the mixed variational inequality or the variational inequality of the second kind. For the applications, numerical methods and other aspects of these mixed variational inequalities.

**II.** If $\varphi$ is an indicator function of a closed and convex set $K$ in a real Hilbert space $H$, that is,

$\varphi(u) = I_K(u) = \begin{cases} 
0, & \text{if } u \in K, \\
+\infty, & \text{otherwise},
\end{cases}$

then problem (3) is equivalent to finding $u \in K$ such that

\[(4)\quad \langle Tu, v - u \rangle \leq 0, \quad \forall v \in K,
\]

which is known as the classical variational inequalities, introduced and studied by Stampacchia [32] in 1964.

We also need the following well-known fundamental results and concepts.

**Definition 2.1.** A nonlinear operator $T : H \to H$ is said to be Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$\|Tu - Tv\| \leq \beta \|u - v\|, \quad \text{for all } u, v \in H.$
Definition 2.2. [2] If $A$ is a maximal monotone operator on $H$, then, for a constant $\rho > 0$, the resolvent operator associated with $A$ is defined by

$$J_A [u] = (I + \rho A)^{-1} [u], \text{ for all } u \in H,$$

where $I$ is the identity operator.

It is known that a monotone operator is maximal if and only if its resolvent operator is defined everywhere. Furthermore, the resolvent operator is a single valued and nonexpansive, that is,

$$\|J_A [u] - J_A [v]\| \leq \|u - v\|, \text{ for all } u, v \in H.$$

Remark 2.3. Since $\partial \varphi$ be a subdifferential of a proper, convex, and lower semicontinuous function $\varphi : H \to R \cup \{+\infty\}$ is a maximal monotone operator, we define by

$$J_\varphi = (I + \rho \partial \varphi)^{-1},$$

the resolvent operator associated with $\partial \varphi$ and $\rho > 0$ is a constant.

Lemma 2.4. [2] For a given $z \in H$, $u \in H$ satisfies the inequality

$$\langle u - z, v - u \rangle + \rho \varphi (v) - \rho \varphi (u) \geq 0, \text{ } \forall v \in R^n,$$

if and only if

$$u = J_\varphi [z],$$

where $J_\varphi$ is the resolvent operator and $\rho > 0$ is a constant.

Using Lemma 2.4, one can show that problem (1) is equivalent to the fixed point problem.

Lemma 2.5. [21] Let $A$ be a maximal monotone operator. Then function $u \in H$, is a solution of the variational inclusion (1) if and only if $u \in H$ satisfies the relation

$$u = J_A [u + \rho Tu],$$

where $J_A = (I + \rho A)^{-1}$ is the resolvent operator and $\rho > 0$ is a constant.

Proof. Let $u \in H$ be a solution of (1). Then

$$0 \in \rho (A (u) - Tu) + u - u$$

$$= u + \rho (A (u)) - (u + \rho Tu)$$

$$= (I + \rho A) u - (u + \rho Tu)$$

$$\iff$$

$$u = (I + \rho A)^{-1} [u + \rho Tu] = J_A [u + \rho Tu],$$

is the desired result.
Lemma 2.5 implies that the variational inclusion (1) is equivalent to the fixed point problem (5). This alternative equivalent formulation is very useful from the numerical and theoretical points of view.

We now define the residue vector $\mathcal{R}(u)$ as:

\begin{equation}
\mathcal{R}(u) = u - J_A [u + \rho Tu].
\end{equation}

It is clear from Lemma 2.5 that problem (1) has a solution $u \in H$, if and only if $u \in H$ is a zero of the equation

\begin{equation}
\mathcal{R}(u) = 0.
\end{equation}

3. Fractional Resolvent Dynamical System

We now propose a dynamical system using residue vector $\mathcal{R}(u)$, defined by the relation (7), related to the problem (1) as:

\begin{equation}
D_\alpha^t u(t) = -\gamma \mathcal{R}(u) = \gamma \{ J_A [u + \rho Tu] - u \}, \quad u_0 \in H,
\end{equation}

where $0 < \alpha < 1$ and $\gamma$ is any constant. The problem (8) is called fractional resolvent dynamical system related to the problem (1).

We now discuss some special cases of problem (8).

I. If $A(\cdot) \equiv \partial \varphi(\cdot)$, the subdifferential of a proper, convex and lower-semicontinuous function $\varphi : H \to R \cup \{\infty\}$, then problem (8) is equivalent to

\begin{equation}
D_\alpha^t u(t) = \gamma \{ J_\varphi [u + \rho Tu] - u \}, \quad u_0 \in H,
\end{equation}

where $0 < \alpha < 1$ and $\gamma$ is any constant. The problem (9) is also called fractional resolvent dynamical system related to the problem (2).

II. If $\varphi$ is an indicator function of a closed and convex set $K$ in a real Hilbert space $H$, then problem (9) is equivalent to

\begin{equation}
D_\alpha^t u(t) = \gamma \{ P_K [u + \rho Tu] - u \}, \quad u_0 \in K,
\end{equation}

where $0 < \alpha < 1$ and $\gamma$ is any constant. The problem (10) is called fractional projected dynamical system related to the problem (4).
III. If $\alpha = 1$, then problem (8) is equivalent to

\[
\frac{du}{dt} = \gamma \left\{ J_A [u + \rho Tu] - u \right\}, \quad u_0 \in H,
\]

where $\gamma$ is any constant. The problem (11) is called resolvent dynamical system related to the problem (1).

For suitable and appropriate choice of operators and parameters, one can propose several new and known dynamical systems.

**Definition 3.1.** [9] The fractional integral (or, Riemann-Liouville integral) with order $\alpha \in \mathbb{R}^+$ of continuous function $u(t)$ is defined as:

\[
I^\alpha_t u(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} u(\tau) \, d\tau, \quad t > t_0.
\]

**Definition 3.2.** [9] The Caputo derivative with order $\alpha \in \mathbb{R}^+$ of continuous function $u(t) \in C^n ([t_0, +\infty), \mathbb{R})$ is defined as:

\[
D^\alpha_t u(t) = I^{n-\alpha}_t u^{(n)}(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{n-\alpha-1} u^{(n)}(\tau) \, d\tau, \quad t > t_0,
\]

where $n$ is positive integer such that $n-1 < \alpha < n$.

**Definition 3.3.** [35] The point $u^*$ is said to be an equilibrium point of the fractional resolvent dynamical system (8), if $u^*$ satisfies the following

\[
J_A [u^*(t) - \rho Tu^*(t)] - u^*(t) = 0.
\]

**Definition 3.4.** [17] The dynamical system (8) is said to converge to the solution set $H^*$ of problem (1) if, irrespective of the initial point, the trajectory of the dynamical system satisfies

\[
\lim_{t \to \infty} \text{dist} (u(t), H^*) = 0,
\]

where

\[
\text{dist} (u, H^*) = \inf_{v \in H^*} \| u - v \|.
\]

Clearly, if the set $H^*$ has a unique point $u^*$, then we have

\[
\lim_{t \to \infty} u(t) = u^*.
\]

The stability of the dynamical system at $u^*$ in the Lyapunov sense, confirms that the dynamical system is also globally asymptotically stable at $u^*$. 
Definition 3.5. [17] The dynamical system is said to be globally exponentially stable with degree \( \eta_1 \) at \( u^* \) if, irrespective of the initial point, the trajectory of the system \( u(t) \) satisfies
\[
\| u(t) - u^* \| \leq \mu_1 \| u(t_0) - u^* \| e^{-\eta_1 (t-t_0)}, \quad \forall t \geq t_0,
\]
where \( \mu_1 > 0 \) and \( \eta_1 > 0 \) are positive constants independent of the initial point. It is clear that globally exponential stability is necessarily globally asymptotically stable and the dynamical system converges arbitrarily fast.

Lemma 3.6. (Gronwall’s Lemma [17]) Let \( u \) and \( v \) be real valued non-negative continuous functions with domain \( \{ t : t \geq t_0 \} \) and let \( \alpha(t) = \alpha(t-t_0) \), where \( \alpha_0 \) is a monotone increasing function. If, for \( t \geq t_0 \),
\[
u(t) \leq \alpha(t) + \int_{t_0}^{t} \alpha(s) v(s) \, ds,
\]
then
\[
u(t) \leq \alpha(t) \cdot \exp \left( \int_{t_0}^{t} v(s) \, ds \right).
\]

Lemma 3.7. [9] Let \( n \) is a positive integer such that \( n - 1 < \alpha < n \). If \( u(t) \in C^n[a,b] \), then
\[
I_{t_0}^\alpha D_{t_0}^\alpha u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!} (t-a)^k.
\]
In particular, if \( 0 < \alpha \leq 1 \) and \( u(t) \in C^1[a,b] \), then
\[
(12) \quad I_{t_0}^\alpha D_{t_0}^\alpha u(t) = u(t) - u(a).
\]

Lemma 3.8. [34] Let \( u(t) \) be a continuous function on \( [0, +\infty) \) and satisfies
\[
(13) \quad D_{t_0}^\alpha u(t) \leq \theta u(t),
\]
where \( 0 < \alpha < 1 \) and \( \theta \) is a constant. Then
\[
u(t) \leq u(0) \cdot \exp \left( \frac{\theta t^\alpha}{\Gamma(\alpha + 1)} \right).
\]

Lemma 3.9. [10, 35] Consider the system
\[
(14) \quad D_{t_0}^\alpha u(t) = \tau(t, x), \quad t > t_0,
\]
with initial condition \( u(t_0) \), where \( 0 < \alpha \leq 1 \) and \( \tau : [t_0, \infty) \times \Omega \rightarrow H \), \( \Omega \subset H \). If \( \tau(t,x) \) satisfies the locally Lipschitz condition with respect to \( x \), then there exists a unique solution of (6) on \( [t_0, \infty) \times \Omega \).

**Lemma 3.10.** [10] For the real valued continuous function \( \tau(t,x) \), defined in (14), we have \( \| I^1_{r} \tau(t,x) \| \leq I^0_{r} \| \tau(t,x) \| \), where \( \alpha \geq 0 \) and \( \| \cdot \| \) denotes an arbitrary norm.

**Theorem 3.11.** Let the operator \( T \) be Lipschitz continuous with constants \( \beta > 0 \). If \( \gamma > 0 \), then for each \( u_0 \in H \), there exists a unique continuous solution \( u(t) \) of the dynamical system (8) with \( u(t_0) = u_0 \) over \( [t_0, \infty) \).

**Proof.** Let 

\[
G(u) = \gamma \{ J_A [u + \rho Tu] - u \}.
\]

To prove that \( G(u) \) is Lipschitz continuous for all \( u \neq v \in H \), we have to consider

\[
\| G(u) - G(v) \| &= \gamma \| \{ J_A [u + \rho Tu] - u \} - \{ J_A [v + \rho Tv] - v \} \|
\leq \gamma \| J_A [u + \rho Tu] - J_A [v + \rho Tv] \| + \gamma \| u - v \|
\leq 2\gamma \| u - v \| + \gamma \rho \| Tu - Tv \|
\leq \gamma (2 + \rho \beta) \| u - v \|,
\]

where we have used Lipschitz continuity of \( T \) with constant \( \beta > 0 \).

This shows that operator \( G(u) \) is a Lipschitz continuous in \( H \). So for each \( u_0 \in H \), there exists a unique and continuous solution \( u(t) \) of the dynamical system (8), defined in an interval \( t_0 \leq t < T_1 \) with the initial condition \( u(t_0) = u_0 \). Let \( [t_0, T_1) \) be its maximal interval of existence. Now we have to show that \( T_1 = \infty \).

Consider,

\[
\| D^1_{r} u(t) \| = \| G(u) \|
= \gamma \| J_A [u + \rho Tu] - u \|
= \gamma \| J_A [u + \rho Tu] - J_A [u] + J_A [u] - J_A [u^*] + J_A [u^*] - u \|
\leq \gamma \| J_A [u + \rho Tu] - J_A [u] \| + \gamma \| J_A [u] - J_A [u^*] \|
\quad + \gamma \| J_A [u^*] \| + \gamma \| u \|
\leq \gamma \| u + \rho Tu - u \| + \gamma \| u - u^* \| + \gamma \| J_A [u^*] \| + \gamma \| u \|
\leq \gamma \rho \beta \| u \| + \gamma (\| u \| + \| u^* \|) + \gamma \| J_A [u^*] \| + \gamma \| u \|
= \gamma (\| u^* \| + \| J_A [u^*] \|) + \gamma (2 + \rho \beta) \| u \|,
\]

where we have used Lipschitz continuity of operator \( T \) with constant \( \beta > 0 \).
Now, taking the fractional integral of (15) over the interval \([t_0, t]\), we have
\[
I_t^\alpha \| D_t^\alpha u (t) \| \leq I_t^\alpha \{ k_1 + k_2 \| u (t) \| \}
\]
\[
= \frac{k_1}{\Gamma (\alpha)} \int_{t_0}^t (t - \tau)^{\alpha - 1} d\tau + \frac{k_2}{\Gamma (\alpha)} \int_{t_0}^t (t - \tau)^{\alpha - 1} \| u (\tau) \| d\tau
\]
\[
= \frac{k_1 (t - t_0)^\alpha}{\Gamma (\alpha + 1)} + \frac{k_2}{\Gamma (\alpha)} \int_{t_0}^t (t - \tau)^{\alpha - 1} \| u (\tau) \| d\tau,
\]
using Lemma 3.7 and Lemma 3.10, we have
\[
\| u (t) \| + \| u (t_0) \| \leq \frac{k_1 (t - t_0)^\alpha}{\Gamma (\alpha + 1)} + \frac{k_2}{\Gamma (\alpha)} \int_{t_0}^t (t - \tau)^{\alpha - 1} \| u (\tau) \| d\tau,
\]
from which by using Lemma 3.6, we have
\[
\| u (t) \| \leq \left\{ \| u (t_0) \| + \frac{k_1 (t - t_0)^\alpha}{\Gamma (\alpha + 1)} \right\} + \frac{k_2}{\Gamma (\alpha)} \int_{t_0}^t (t - \tau)^{\alpha - 1} \| u (\tau) \| d\tau
\]
\[
\leq \left\{ \| u (t_0) \| + \frac{k_1 (t - t_0)^\alpha}{\Gamma (\alpha + 1)} \right\} \exp \left\{ \frac{k_2 (t - t_0)^\alpha}{\Gamma (\alpha + 1)} \right\},
\]
where
\[
k_1 = \gamma \{ \| u^* \| + \| J_A [u^*] \| \}
\]
\[
k_2 = \gamma (2 + \rho \beta) > 0.
\]
This shows that the solution is bounded on \([t_0, \infty)\).

We now show that the trajectory of the solution of the dynamical system (8) converges \(\alpha\)-exponentially to the unique solution of problem (1).

**Theorem 3.12.** Let operator \(T : H \to H\) be Lipschitz continuous with constant \(\beta > 0\). If \(\gamma < 0\) is a constant, then dynamical system (8) converges \(\alpha\)-exponentially to the unique solution of problem (1).

**Proof.** Let \(u (t)\) and \(v (t)\) be any two solutions of dynamical system (8) with initial values \(u (0) = u_0\) and \(v (0) = v_0\), respectively.

Let
\[
e (t) = u (t) - v (t),
\]
then $e(0) \neq 0$ and taking the fractional derivative of above equation, we have

$$D_0^\alpha e(t) = D_0^\alpha u(t) - D_0^\alpha v(t)
= \gamma \{ J_A [u(t) + \rho Tu(t)] - u(t) \} - \gamma \{ J_A [v(t) + \rho Tv(t)] - v(t) \}
\leq \gamma \{ J_A [u(t) + \rho Tu(t)] - J_A [v(t) + \rho Tv(t)] \} - \gamma e(t),$$

where $0 < \alpha < 1$, $t \geq 0$. From Theorem 3.11, $u(t)$ and $v(t)$ are uniquely determined solutions. Therefore $e(t)$ is the uniquely determined solution of error system (16) with initial value $e(0) = e_0$.

We claim that if $e(0) > 0$, then $e(t) \geq 0$ for $t \geq 0$, if $e(0) < 0$, then $e(t) \leq 0$ for $t \geq 0$. In fact, if $e(0) > 0$, there exists $t_0$, such that $e(t) < 0$ for $t \geq t_0$, so there must be $0 < t_0 < t_1$ such that $e(t_0) = 0$. It means that dynamical system (8) has two different solutions with initial value $t_0$ to $e(t_0) \leq 0$ for $t \geq t_0$, which contradicts to Theorem 3.11. In a similar way, we can prove that if $e(0) < 0$, then $e(t_0) \leq 0$ for $t \geq 0$.

So, we can have

$$D_0^\alpha G(t) = D_0^\alpha \| e(t) \|
= sgn(e(t)) (D_0^\alpha e(t))
= \gamma sgn(e(t)) \{ J_A [u(t) + \rho Tu(t)] - J_A [v(t) + \rho Tv(t)] \}
\leq \gamma \| J_A [u(t) + \rho Tu(t)] - J_A [v(t) + \rho Tv(t)] \| + \gamma \| e(t) \|
\leq \gamma \| u(t) - v(t) + \rho (Tu(t) - Tv(t)) \| + \gamma \| e(t) \|
\leq \gamma \{ u(t) - v(t) \} + \gamma \rho \| u(t) - v(t) \| + \gamma \| e(t) \|
= \gamma (2 + \rho \beta) \| e(t) \|,$$

which implies

$$D_0^\alpha G(t) \leq \gamma (2 + \rho \beta) G(t)
= k_3 G(t),$$

thus by using Lemma 3.8, we have

$$G(t) \leq G(0) \exp \left\{ \frac{k_3 t^\alpha}{\Gamma(\alpha + 1)} \right\},$$

where

$$k_3 = \gamma (2 + \rho \beta) < 0.$$
which shows that the dynamical system (8) is $\alpha$-exponentially stable.

4. Resolvent Equation Dynamical System

In this section, we consider the problem of solving the resolvent equations. It is shown that the variational inclusions (1) are equivalent to the resolvent equations. This alternative equivalent formulation is used to suggest and investigate a class of iterative methods for solving the variational inclusions (1).

We now consider the problem of solving the resolvent equations. Let $R_A = I - J_A$, where $J_A$ is the resolvent operator and $I$ is the identity operator. For given nonlinear operators $T$, $A$ consider the problem of finding $z \in H$ such that

\begin{equation}
TJ_A [z] - \rho^{-1} R_A [z] = 0.
\end{equation}

Equations of the type (17) are called resolvent equations associated with the problem (1).

We now discuss some special cases of problem (17).

I. If $A (\cdot) \equiv \partial \varphi (\cdot)$, the subdifferential of a proper, convex and lower-semicontinuous function $\varphi : H \to R \cup \{\infty\}$, then problem (17) is equivalent to

\begin{equation}
TJ_\varphi [z] - \rho^{-1} R_\varphi [z] = 0.
\end{equation}

This problem is known resolvent equation related to the problems (2) and (3).

II. If $\varphi$ is an indicator function of a closed and convex set $K$ in a real Hilbert space $H$, then problem (17) is equivalent to

\begin{equation}
TP_K [z] - \rho^{-1} Q_K [z] = 0,
\end{equation}

which is called Wiener-Hopf equation related to the problem (4).

This problem introduced and studied by Shi [30] in connection with variational inequalities. This shows that the Wiener-Hopf equations are a special case of the resolvent equations. The resolvent equations technique has been used to study and develop several iterative methods for solving various type of variational inequalities and inclusions problems, see [21, 22, 23].
Using Lemma 2.5, one can show that the variational inclusions (1) are equivalent to the resolvent equations (17).

**Lemma 4.1.** [21] The variational inclusion (1) has a solution \( u \in H \), if and only if, the resolvent equations (17) have a solution \( z \in H \), provided

\[
(20) \quad u = J_A[z],
\]

\[
(21) \quad z = u + \rho Tu,
\]

where \( \rho > 0 \) is a constant.

Using Lemma 4.1, the resolvent equation (17) can be written as

\[
(22) \quad u + \rho Tu - J_A [u + \rho Tu] - \rho T J_A [u + \rho Tu] = 0,
\]

from which we have

\[
(23) \quad u = J_A [u + \rho Tu] + \rho T J_A [u + \rho Tu] - \rho Tu.
\]

Thus it is clear from Lemma 4.1 that \( u \in H \) is a solution of problem (1), if and only if, \( u \in H \) satisfies the equation (22).

We now propose a new fractional dynamical system:

\[
(24) \quad D^\alpha_t u(t) = \gamma \{ J_A [u + \rho Tu] + \rho T J_A [u + \rho Tu] - \rho Tu - u \},
\]

where \( u(t_0) = u_0 \in H \), \( 0 < \alpha < 1 \) and \( \gamma \) is any constant. This problem is called fractional resolvent equation dynamical system related to the problem (1).

We now discuss some special cases of problem (24).

**I.** If \( A(\cdot) = \partial \varphi(\cdot) \), the subdifferential of a proper, convex and lower-semicontinuous function \( \varphi : H \to \mathbb{R} \cup \{\infty\} \), then problem (24) is equivalent to

\[
(25) \quad D^\alpha_t u(t) = \gamma \{ J_\varphi [u + \rho Tu] + \rho T J_\varphi [u + \rho Tu] - \rho Tu - u \},
\]

where \( u(t_0) = u_0 \in H \), \( 0 < \alpha < 1 \) and \( \gamma \) is any constant. The problem (25) is called fractional resolvent equation dynamical system related to the problem (2).

**II.** If \( \varphi \) is an indicator function of a closed and convex set \( K \) in a real Hilbert space \( H \), then problem (9) is equivalent to

\[
(26) \quad D^\alpha_t u(t) = \gamma \{ P_K [u + \rho Tu] + \rho T P_K [u + \rho Tu] - \rho Tu - u \},
\]
where \( u(t_0) = u_0 \in K, \ 0 < \alpha < 1 \) and \( \gamma \) is any constant. The problem (26) is called fractional projected dynamical system related to the problem (4).

III. If \( \alpha = 1 \), then problem (24) is equivalent to

\[
\frac{du}{dt} = \gamma \{ J_A [u + \rho Tu] + \rho T J_A [u + \rho Tu] - \rho Tu - u \},
\]

where \( u(t_0) = u_0 \in H \) and \( \gamma \) is any constant. The problem (27) is called resolvent equation dynamical system related to the problem (1).

For different suitable and appropriate choice of operators and spaces one can suggest several known and new dynamical systems.

We now study the main properties of the proposed problem (24) and analyze the global stability of the systems. First of all, we discuss the existence and uniqueness of the dynamical system (24) and this is the main motivation of our next result.

Theorem 4.2. Let the nonlinear operator \( T \) be Lipschitz continuous with constant \( \beta > 0 \). If \( \gamma > 0 \), then for each \( u_0 \in H \), there exists a unique continuous solution \( u(t) \) of the dynamical system (24) with \( u(t_0) = u_0 \) over \([t_0, \infty)\).

Proof. Let

\[
\mathcal{G}(u) = \gamma \{ J_A [u + \rho Tu] + \rho T J_A [u + \rho Tu] - \rho Tu - u \}.
\]

To prove that \( \mathcal{G}(u) \) is Lipschitz continuous for all \( u \neq v \in H \), we consider

\[
\| \mathcal{G}(u) - \mathcal{G}(v) \| = \gamma \| \{ J_A [u + \rho Tu] + \rho T J_A [u + \rho Tu] - \rho Tu - u \}
\]

\[
- \{ J_A [v + \rho Tv] + \rho T J_A [v + \rho Tv] - \rho Tv - v \} \|
\]

\[
\leq \gamma \| J_A [u + \rho Tu] - J_A [v + \rho Tv] \|
\]

\[
+ \gamma \rho \| T J_A [u + \rho Tu] - T J_A [v + \rho Tv] \|
\]

\[
+ \gamma \rho \| Tu - Tv \| + \gamma \| u - v \|
\]

\[
\leq \gamma (1 + \rho \beta) \| J_A [u + \rho Tu] - J_A [v + \rho Tv] \|
\]

\[
+ \gamma (1 + \rho \beta) \| u - v \|
\]

\[
\leq 2 \gamma (1 + \rho \beta) \| u - v \| + \gamma \rho (1 + \rho \beta) \| Tu - Tv \|
\]

\[
\leq \gamma (1 + \rho \beta) (2 + \rho \beta) \| u - v \|,
\]

where we have used the Lipschitz continuity of \( T \) with constant \( \beta > 0 \).
This implies that operator $G(u)$ is Lipschitz continuous in $H$. So for each $u_0 \in H$, there exists a unique and continuous solution $u(t)$ of the dynamical system (24), defined in an interval $t_0 \leq t < T_1$ with the initial condition $u(t_0) = u_0$. Let $[t_0, T_1)$ be its maximal interval of existence. Now we have to show that $T_1 = \infty$.

Consider,

$\|D^\alpha_t u(t)\| = \|G(u)\| = \gamma \|J_A [u + \rho Tu] - u\| \leq \gamma \|J_A [u + \rho Tu] - u\| + \gamma \rho \|TJ_A [g(u) + \rho Tu] - Tu\| \leq \gamma (1 + \rho \beta) \|J_A [u + \rho Tu] - u\|$

\begin{equation}
\leq \gamma (1 + \rho \beta) \{\|u^*\| + \|J_A [u^*]\|\}
\end{equation}

where we have used the Lipschitz continuity of $T$ with constant $\beta > 0$ respectively.

By taking the fractional integral of (28) over the interval $[t_0, t]$, and then using Lemma 3.6, Lemma 3.7 and Lemma 3.10, we obtain the following result

$\|u(t)\| \leq \left\{\|u(t_0)\| + \frac{k_5 (t-t_0)^\alpha}{\Gamma(\alpha + 1)}\right\} + \frac{k_6}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} \|u(\tau)\| \, d\tau$

$\leq \left\{\|u(t_0)\| + \frac{k_5 (t-t_0)^\alpha}{\Gamma(\alpha + 1)}\right\} \exp \left\{\frac{k_6 (t-t_0)^\alpha}{\Gamma(\alpha + 1)}\right\}$,

where

$k_5 = \gamma (1 + \rho \beta) \{\|u^*\| + \|J_A [u^*]\|\}$,

$k_6 = \gamma (1 + \rho \beta) (2 + \rho \beta) > 0$.

This shows that the solution is bounded on $[t_0, \infty)$.

We now show that the trajectory of the solution of the dynamical system (24) converges $\alpha$-exponentially to the unique solution of problem (1).

**Theorem 4.3.** Let operator $T$ be Lipschitz continuous with constants $\beta > 0$. If $\gamma < 0$, then the dynamical system (24) converges $\alpha$-exponentially to the unique solution of problem (1).

**Proof.** Let $u(t)$ and $v(t)$ be any two solutions of dynamical system (24) with initial values $u(0) = u_0$ and $v(0) = v_0$, respectively.
Let
\[ e(t) = u(t) - v(t), \]
then \( e(0) \neq 0 \) and taking the fractional derivative of above equation, we have

\[
D_t^\alpha e(t) = D_t^\alpha u(t) - D_t^\alpha v(t)
\]

\[
= \gamma \{ J_A [u + \rho Tu] + \rho T J_A [u + \rho Tu] - \rho Tu - u \} - \gamma \{ J_A [v + \rho Tv] + \rho T J_A [v + \rho Tv] - \rho Tv - v \}
\]

\[
= \gamma \{ J_A [u + \rho Tu] - J_A [v + \rho Tv] \} + \gamma \rho \{ T J_A [u + \rho Tu] - T J_A [v + \rho Tv] \}
\]

\[
- \gamma \rho \{ T u - T v \} - e(t),
\]

(29)

where \( 0 < \alpha < 1, t \geq 0 \). From Theorem 4.2, \( u(t) \) and \( v(t) \) are uniquely determined solutions. Therefore \( e(t) \) is the uniquely determined solution of error system (29) with initial value \( e(0) = e_0 \).

We claim that if \( e(0) > 0 \), then \( e(t) \geq 0 \) for \( t \geq 0 \), if \( e(0) < 0 \), then \( e(t) \leq 0 \) for \( t \geq 0 \). In fact, if \( e(0) > 0 \), there exists \( t_0 \), such that \( e(t) < 0 \) for \( t \geq t_0 \), so there must be \( 0 < t_0 < t_1 \) such that \( e(t_0) = 0 \). It means that dynamical system (24) has two different solutions with initial value \( t_0 \) to \( e(t_0) \leq 0 \) for \( t \geq t_0 \), which contradicts to Theorem 4.2. In a similar way, we can prove that if \( e(0) < 0 \), then \( e(t_0) \leq 0 \) for \( t \geq 0 \).

So, we can have

\[
D_t^\alpha \mathcal{G}(t) = D_t^\alpha \| e(t) \|
\]

\[
= sgn(e(t)) \langle D_t^\alpha e(t) \rangle
\]

\[
= \gamma sgn(e(t)) \{ J_A [u + \rho Tu] - J_A [v + \rho Tv] \} + \gamma \rho sgn(e(t)) \{ T J_A [u + \rho Tu] - T J_A [v + \rho Tv] \}
\]

\[
- \gamma \rho sgn(e(t)) \{ T u - T v \} - \gamma \| e(t) \|
\]

\[
\leq \gamma \| J_A [u + \rho Tu] - J_A [v + \rho Tv] \|
\]

\[
+ \gamma \rho \| T J_A [u + \rho Tu] - T J_A [v + \rho Tv] \|
\]

\[
+ \gamma \rho \| T u - T v \| + \gamma \| e(t) \|
\]

\[
\leq \gamma (1 + \rho \beta) \| J_A [u + \rho Tu] - J_A [v + \rho Tv] \| + \gamma (1 + \rho \beta) \| e(t) \|
\]

\[
\leq \gamma (1 + \rho \beta) \| u - v + \rho (Tu - Tv) \| + \gamma (1 + \rho \beta) \| e(t) \|
\]

\[
\leq \gamma (1 + \rho \beta) \| u - v \| + \gamma \rho \beta \| u - v \| + \gamma (1 + \rho \beta) \| e(t) \|
\]

\[
= \gamma (1 + \rho \beta) (2 + \rho \beta) \| e(t) \|,
\]
which implies

\[ D^{\alpha}_t \mathcal{G}(t) \leq (1 + \rho \beta)(2 + \rho \beta) \mathcal{G}(t) = k_5 \mathcal{G}(t), \]

thus by using Lemma 3.8, we have

\[ \mathcal{G}(t) \leq \mathcal{G}(0) \exp \left\{ \frac{k_5 t^\alpha}{\Gamma(\alpha + 1)} \right\}, \]

where

\[ k_7 = (1 + \rho \beta)(2 + \rho \beta) < 0. \]

Let \( k_7 = -k_8 \), where \( k_8 \) is a positive constant. Then

\[ \mathcal{G}(t) \leq \mathcal{G}(0) \exp \left( \frac{-k_8 t^\alpha}{\Gamma(\alpha + 1)} \right), \]

which shows that the dynamical system (8) is \( \alpha \)-exponentially stable. \( \square \)

5. Conclusion

In this paper, we have introduced and studied fractional resolvent dynamical systems and fractional resolvent equation dynamical systems. Using essentially the resolvent technique, we have established the equivalence between the variational inclusions and the resolvent equations. This alternate formulation is used to suggest and analyze fractional dynamical systems related to the variational inclusions involving difference of monotone operators. We have proved that these dynamical systems converge \( \alpha \)-exponentially to the unique solution of variational inclusion problems under some suitable conditions. The suggested dynamical systems can be used in designing recurrent neural networks for solving variational inclusions and related optimization problems. Some special cases are also discussed.

Competing interests: The authors declare that they have no competing interests.

Author’s contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

Fractional dynamical systems for variational inclusions 223


Awais Gul Khan
Department of Mathematics, Government College University Faisalabad,
Allama Iqbal Road, Faisalabad, Pakistan.
E-mail: awaisgulkhan@gmail.com

Muhammad Aslam Noor
Department of Mathematics, COMSATS University Islamabad,
Park Road, Islamabad, Pakistan.
E-mail: noormaslam@gmail.com

Khalida Inayat Noor
Department of Mathematics, COMSATS University Islamabad,
Park Road, Islamabad, Pakistan.
E-mail: khalidan@gmail.com