Stabilization of Co Semigroups in Infinite Dimensional Systems by a Compact Linear Feedback via the Steady State Riccati Equation

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ABSTRACT

Stability of Co Semigroups perturbed via the steady state Riccati equation (SSRE) is studied. We consider an infinite dimensional system : $\dot{x} = Ax + Bu$, in $\mathscr{D}(A)$, domain of A, where A is the infinitesimal generator of a Co semigroup $[T(t),\ t\geq 0]$ in H.

If the original Co semigroup $[T(t),\ t\geq 0]$ has a lower bound : $\|T(t)x\| \geq k\|x\|$, for all x in H, $t\geq 0$ and $k>0$, then the perturbed Co semigroup via the SSRE, where the feedback operator B is compact, cannot be exponentially stable. Physical interpretation of this result is as follows : in real applications, a finite number of actuators are available, therefore the operator B is compact.

When the original system is inherently unstable, that is, has an infinite number of unstable modes, the perturbed system via the SSRE cannot be stable with a uniform decay rate.

1. Introduction

In this paper we will deal with stability of stronly continuous (i.e., Co) semigroups of bounded linear operators on Hilbert space H. The inner product and norm are denoted by $[\cdot,\cdot]$ and by $\|\cdot\|$, respectively.

Now consider the linear system (A, B) such that

$$\dot{x}=Ax+Bu,$$

where A is the infinitesimal generator of a Co semigroup $[T(t),\ t\geq 0]$, and B is a bounded linear operator from another Hilbert space U to H.

The system (A, B) is called to be "stable" is for each x in H : $T(t)x \to 0$, $t\to\infty$ in a certain sense which will be discussed later. Now if the system (A, B) is not stable and if there exists a bounded linear state feedback operator F : $H\to U$, $u=Fx$, such that $A+BF$ generates a stable semigroup $[S(t), t\geq 0]$, then the system (A, B) is said to be state feedback stabilizable.

We study, in this paper, stability of a linear distributed parameter system by means of a state feedback involving a solution of the steady state Riccati equation (SSRE) which results from the linear quadratic regulator problem, or from the Kalman filtering problem. Specifically we investigate stability of a semigroup $[S(t), t\geq 0]$ whose generator is $A-BB^*P$, where P is bounded linear, self-adjoint and nonnegative, $P\geq 0$. P satisfies the SSRE:

$$[Px,Ax]+[Ax,Px]+[Rx,x]-[PBB^*Px,x]=0$$

for x in the domain $\mathscr{D}(A)$ of A. Here R is also a linear bounded, self-adjoint and nonnegative, $R\geq 0$, operator.

It is natural to extend the well-known results of the finite dimensional control system theory to the infinite dimensions. Most of the works to date emphasize exact controllability and exponential stabilizability [4], [12], which, of course, are generalization of the finite dimensional results of Lyapunov [7], Kalman [8] and Wonham [11].

In finite dimensions, it is well-known that controllability implies stabilizatility. In infinite dimensional systems, these results of finite dimensional systems can not be easily extended. We cannot have exact controllability when the semigroup $[T(t),\ t\geq 0]$ or the operator B is compact [10]. Also we cannot have exponential stability by compact feedback in the case of a linear oscillatory system [6].

These difficulties motivated us to study the limitations of exponential stabilizability of infinite dimensional systems. In practice, we use a finite number of control elements to stabilize the unstable systems according to the control law based on the observed state from a finite number of sensors in the systems. This implies that the operator B is compact. The original Co semigroup which we deal with in this paper is a semigroup $[T(t),\ t\geq 0]$ with a undamping oscillatory part, that is, the norm $\|T(t)x\|$ has a lower bound such that $\|T(t)x\|\geq k\|x\|$, $k>0$ for all x in H, $t\geq 0$.

In Section 2 we will introduce basic notions of infinite dimensional systems. We deal with generation of the perturbed semigroup $[S(t),\ t\geq 0]$ via the SSRE, and stability of $[S(t),\ t\geq 0]$ is considered in the case that the original semigroup $[T(t),\ t\geq 0]$ has a lower bound in Section 3. Also an interesting example, which shows that exponential stabilizability can be achieved when the unstable states are finite, is discussed. In the last Section the limitations of exponential stabilizability and physical interpretation of the results of this paper is considered.

2. Preliminary Results of Infinite Dimensional Systems

We begin with various notions of positivity of linear operators.

Definition 2.1 A linear bounded self-adjoint opeator L on a complex Hilbert space is said to be

i) nonnegative, written by $L\geq 0$, if and only if $[Lx,x]\geq 0$, for each x in H,
ii) positive, written by $L>0$, if and only if $[Lx,x]>0$, for each x in H,

iii) strictly positive, written by $L > \alpha I, \alpha > 0$ such that $[Lx, x] \geq \alpha \|x\|^2$, for each x in H

The notion of controllability can be extended to the infinite dimensions as follows [12] :

Theorem 2.1 The following two statements are equivlent.

i) The system (A, B) is exactly controllable.
ii) There exists $t > 0$ and $\alpha > 0$ such that $\int_0^t \|B^* T^*(\sigma)x\|^2 d\sigma \geq \alpha \|x\|^2$, for x in H.

In many applications, the operator B is compact. Hence the part ii) of the above theorem does not hold. Now we have [1], [10]:

Theorem 2.2 If the semigroup $[T(t), t \geq 0]$ or the operator B is compact, the system (A, B) cannot be exactly controllable on H.

The next best property to exact controllability is when the controllable subspace is dense in H.

Theorem 2.3 A necessary and sufficient condition for the system (A, B) to be approximately controllable is

$$\int_0^t \|B^* T^*(\sigma)x\|^2 d\sigma = 0 \text{ implies that } x = 0, \text{ for any } t > 0.$$

Since observability is the dual of controllability, if follows that

Definition 2.2 The pair (C, A) is said to be

i) exactly observable if and only if, for some $t > 0, \alpha > 0, \int_0^t \|CT(\sigma)x\|^2 d\sigma \geq \alpha \|x\|^2$, for x in H,
ii) approximately observable if and only if, for $t > 0$, $CT(t)x = 0$ implies that $x = 0$.

As in the finite dimensional case, it follows that (A, B) exact (approximate) controllability $\Longleftrightarrow$ (B^*, A^*) exact (approximate) observability.

Now the notion of stability of finite dimensional case can be generalized to infinite dimensional systems :

Definition 2.3 A Co semigroup $[T(t), t \geq 0]$ is said to be exponentially stable if there exist constants $M \geq 1, \alpha > 0$ such that

$$\|T(t)\| \leq Me^{-\alpha t}, \quad t \geq 0$$

Definition 2.4 A Co semigroup $[T(t), t \geq 0]$ is said to be strongly stable if for each x in H

$$\lim_{t \to \infty} \|T(t)x\| = 0.$$

Definition 2.5 A Co semigroup $[T(t), t \geq 0]$ is said to be weakly stable, if for x, y in H

$$\lim_{t \to \infty} [T(t)x, y] = 0.$$

We here consider examples of strongly and weakly stable semigroups [2].

Let $[T(t), t \geq 0]$ be the left shift operator such that, for x in $H = L_2[0, \infty)$.

$$T(t)x(\xi) = \begin{cases} x(\xi + t), & \text{if } \xi + t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

where inner product is defined by

$$[x, y] = \int_0^\infty x \cdot \bar{y} d\xi; \text{ for } x, y \text{ in } L_2[0, \infty).$$

Then $\|T(t)x\|^2 = \int_0^\infty \|x(t+\xi)\|^2 d\xi = \int_t^\infty \|x(\eta)\|^2 d\eta \to 0$ as $t \to \infty$.

However $\|T(t)\| = 1$.

On the other hand for the right shift operator such that, for x in $H = L_2[0, \infty)$.

$$T(t)x(\xi) = \begin{cases} x(\xi - t), & \text{if } \xi - t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

We then have $\|T(t)x\|^2 = \|x\|^2$, but $\lim_{t \to \infty} [T(t)x, y] = 0$ for x, y in H.

Several results are shown for the semigroups which have special properties [5].

Theorem 2.4 If a Co semigroup $[T(t), t \geq 0]$ is compact for some $t_0 > 0$, then $[T(t), t \geq 0]$ is exponentially stable if and only if it is weakly stable.

The standard results of finite dimensional systems can be generalized as follows [4], [12].

Theorme 2.5 Suppose the system (A, B) is exactly controllable, then it is exponentially stabilizable.

Theorem 2.6 A necessary and sufficient condition for the semigroup $[T(t), t \geq 0]$ to be exponentially stable on H is that there exists a bounded linear, self-adjoint and nonnegative operator P on H such that

$$2Re[PAx, x] = -\|x\|^2, \text{ for each } x \text{ in } \mathcal{D}(A).$$

Also an interesting results can be derived from the above theorem [4].

Theorem 2.7 A Co semigroup $[T(t), t \geq 0]$ is exponentially stable if and only if

$$\int_0^\infty \|T(t)x\|^2 dt < \infty, \text{ for each } x \text{ in } H.$$

Now consider a system (A, B) with a cost functional:

$$J(x, u) = \int_0^\infty \{[Rx(t), x(t)] + [u(t), u(t)]\} dt,$$

where R is a bounded linear self-adjoint and nonnegative operator on H.

As in the finite dimensional case, the optimal feedback solution $u(t) = -B^* Px(t)$, $t \geq 0$, can be obtained from the "inner product" steady state Riccati equation (SSRE):

$$[Px, Ax] + [Ax, Px] + [Rx, x] - [PBB^* Px, x] = 0$$

for x in $\mathcal{D}(A)$, where the solution P, which is assumed to exist throughout this paper, is nonnegative, self-adjoint operator on H.

Wonham's results [11] in finite dimensoinal systems can be generalized as follows [12].

Theorme 2.8 If (A, B) and (A^*, R) are exponentially stabilizable, then there exists a bounded linear, self-adjoint and nonnegative solution P of the SSRE, and the semigroup $[S(t),\ t \geq 0]$ with generator $A - BB^*P$ is exponentially stable

3. Stability of Co Semigroups Perturbed via the SSRE

We now show generation of the semigroup $[S(t),\ t \geq 0]$ with the generator $A - BB^*P$ from the SSRE.

Consider the SSRE: for x in $\mathcal{D}(A)$

$$[Px, Ax] + [Ax, Px] + [Rx, x] - [PBB^*Px, x] = 0 \quad (3.1)$$

Substracting $2[PBB^*Px, x]$ on both sides of (3.1), we obtain, for x in $\mathcal{D}(A)$,

$$2Re[P(A - BB^*P)x, x] = -[Rx, x] - \|B^*Px\|^2 \leq 0 \quad (3.2)$$

Since $S(t)x$ is in $\mathcal{D}(A)$, it follows that, for x in $\mathcal{D}(A)$,

$$\frac{d}{dt}[PS(t)x, S(t)x] = -[RS(t)x, S(t)x] - \|B^*PS(t)x\|^2 \quad (3.3)$$

Since $\mathcal{D}(A)$ is dense in H, integrating both sides of (3.3) we have, for x in H.

$$[PS(t)x, S(t)x] - [Px, x]$$
$$= -\int_0^t [RS(\sigma)x, S(\sigma)x] - \|B^*PS(\sigma)x\|^2 d\sigma,\ t \geq 0 \quad (3.4)$$

Theorem 2.7 are now applied to the semigroup generated from the SSRE. We have

Theorem 3.1 Let $P \geq 0$ be a solution of the SSRE where R is taken to be strictly positive, then the semigroup $[S(t),\ t \geq 0]$ with generator $A - BB^*P$ is exponentially stable.

Proof We have from the SSRE, for x in $\mathcal{D}(A)$

$$2Re[P(A - BB^*P)x, x] = -[Rx, x] - \|B^*Px\|^2$$

The existence of a solution P of the SSRE implies that, for x in H,

$$\int_0^\infty [RS(t)x, s(t)x]dt \leq [Px, x] < \infty,$$

and since R is strictly positive, we conclude that

$$\int_0^\infty \|S(t)x\|^2 dt < \infty, \text{ for } x \text{ in } H$$

Therefore $[S(t),\ t \geq 0]$ is exponentially stable by Theorem 2.7.

Now we consider the case that the original semigroup $[T(t),\ t \geq 0]$ has a lower bound such that $\|T(t)x\| \geq k\|x\|,\ k > 0$, for x in H

First we have [9].

Lemma 3.1 Let $\{\phi_n\}$, $n = 1, 2, \cdots$ be an orthonormal sequence in the Hibert space H, then the linear operator K mapping H to itself defined by

$$Kx = \sum_{n=1}^\infty \alpha_n \phi_n [\phi_n, x], \text{ for } x \text{ in } H.$$

is compact if and only if

$$\lim_{n \to \infty} |\alpha_n| = 0$$

Therefore if the operator B is compact, then B is nonnegative $B \geq 0$, and $inf\,\|Bx\| = 0$

Now we show the follwoing theorem.

Theorme 3.2 For a system (A, B) if the semigroup $[T(t),\ t \geq 0]$ with generator A has a lower bound such that

$$\|T(t)x\| \geq k\|x\|,\ k > 0, \text{ for } t \geq 0,\ x \text{ in } H,$$

then the system cannot be exponentially stabilized by a comapct feedback $F = -B^*P$, where P is a bounded linear, self-adjoint and nonnegative solution of the SSRE.

Proof. Since we have, for x in H [1],

$$S(t) = T(t)x - \int_0^t T(t - \sigma)BB^*PS(\sigma)d\sigma, \quad (3.5)$$

We can obtain by the triangular inequality and Schwartz inequality

$$\|S(t)x\| \geq \|T(t)x\| - \int_0^t \|T(t-\sigma)\|\|BB^*PS(\sigma)x\|d\sigma,\ t \geq 0 \quad (3.6)$$

Let for an arbitary $\bar{x}$ in H

$$\int_0^t \|T(t-\sigma)\|\|BB^*PS(\sigma)\bar{x}\|d\sigma,\ t \geq 0$$
$$= \inf_x \int_0^t \|T(t-\sigma)\|\,\|BB^*PS(\sigma)x\|d\sigma, \quad (3.7)$$

then

$$\|S(t)\bar{x}\| \geq \|T(t)\bar{x}\| - \int_0^t \|T(t-\sigma)\|\|BB.PS(\sigma)\bar{x}\|d\sigma$$
$$\sup_x \|S(t)x\| \geq \|S(t)\bar{x}\| \geq \|T(t)\bar{x}\|$$
$$- \inf_x \int_0^t \|T(t-\sigma)\|\|BB^*PS(\sigma)x\|d\sigma$$
$$\geq \inf_x \|T(t)x\|$$
$$- \inf_x \int_0^t \|T(t-\sigma)\|\|BB^*PS(\sigma)x\|d\sigma \quad (3.9)$$

By compactness of B^*P, BB^*P is also compact, hence

$$\inf_x \|BB^*PS(\sigma)x\| = 0,\ \sigma \geq 0$$

It thus follows from (3.9) that

$$\|S(t)\| = \sup_x \frac{\|S(t)x\|}{\|x\|} \geq k > 0, \text{ for } t \geq 0$$

Therefore $[S(t),\ t \geq 0]$ cannot be exponentially stable

Remark 1 The above theorem is more generalization of the Gibson's result [6] where $\|T(t)x\| = \|x\|$.

Remark 2 If the original semigroup $[T(t),\ t\geq 0]$ has a lower bound: $\|T(t)x\|\geq k\|x\|,\ t\geq 0$, for x in H, then the system (A, B) has an infinite number of unstable states.

Next we consider an example which shows many interesting properties we have discussed before.

Example 2.1 [Heat equation : bounded domain].
Consider the system :

$$\frac{\partial x}{\partial t}=\frac{\partial^2 x}{\partial \xi^2}+Bu(t,\xi),\quad 0\leq \xi\leq 2\pi,\quad t\geq 0.$$

Let $H=L_2[0, 2\pi]$, and the boundary conditions are

$$x(0)=x(2\pi),\quad x'(0)=x'(2\pi).$$

We have $A=\frac{\partial^2}{\partial\xi^2}$, and $\mathcal{D}(A)=\{x$ *in* $H: x, x'$: absolutely continuous $: x', x''$ *in* H ; and $x(0)=x(2\pi), x'(0)=x'(2\pi)$
We observe that for x in $\mathcal{D}(A)$,

$$[Ax, x]=\frac{1}{2\pi}\int_0^{2\pi} x''(\xi)\overline{x(\xi)}d\xi$$
$$=-\frac{1}{2\pi}\int_0^{2\pi}|x'(\xi)|^2 d\xi$$
$$=[x, Ax]\leq 0$$

Hence A is self-adjoint and dissipative, and it generates a contraction semigroup $[T(t),\ t\geq 0]$:

$$T(t)x=\sum_{n=-\infty}^{\infty} e^{-n^2 t}\phi_n[\phi_n, x],\quad t\geq 0 \text{ and for } x \text{ in } H.$$

where $\phi_n=e^{in\xi}$, $n=0,\pm 1,\pm 2,\cdots$, is an orthogonal sequence in H.
Now it is easy to see that

$$\|T(t)\phi_0\|=\|\phi_0\| \ :\ \phi_0 \text{ is unstable}$$
$$\|T(t)\phi_n\|<\|\phi_n\| \text{ for } n\neq 0,\ n=\pm 1,\pm 2,\cdots,$$

Let

$$Bx=\sum_{n=-\infty}^{\infty}\frac{1}{n^2+1}\phi_n[\phi_n, x]$$
$$Px=\sum_{n=-\infty}^{\infty}\frac{1}{n^2+1}\phi_n[\phi_n, x]$$

and $Rx=\sum_{n=-\infty}^{\infty}(2-\frac{2}{n^2+1}+\frac{1}{(n^2+1)^4})\phi_n[\phi_n, x]$ for x in H, then P satisfies the SSRE.

Since $\lim_{n\to\pm\infty}\left|\frac{1}{n^2+1}\right|=0$, B and P are compact. Now

$$[Rx, x]=\sum_{n=-\infty}^{\infty}(2-\frac{2}{n^2+1}+\frac{1}{(n^2+1)^4})|[\phi_n, x]|^2$$
$$\geq\sum_{n=-\infty}^{\infty}|[\phi_n, x]|^2=\|x\|^2, \text{ for } x \text{ in } H.$$

R is strictly positive, that is, R is not compact.
Now consider, for some $\varepsilon>0$

$$\int_0^{\varepsilon}[RT(t)x, T(t)x]dt$$
$$=\int_0^{\varepsilon}[\sum_{n=-\infty}^{\infty}(2-\frac{2}{n^2+1}+\frac{1}{(n^2+1)^4})e^{-n^2 t}\phi_n[\phi_n, x],$$
$$\sum_{n=-\infty}^{\infty}e^{-n^2 t}\phi_n[\phi_n, x]]dt$$
$$=\sum_{n=-\infty}^{\infty}\int_0^{\varepsilon}(2-\frac{2}{n^2+1}+\frac{1}{(n^2+1)^4})e^{-2n^2 t}\ |[\phi_n, x]|^2 dt$$
$$=\varepsilon|[\phi_0, x]|^2+\sum_{n=-\infty}^{\infty}(2-\frac{2}{n^2+1}+\frac{1}{(n^2+1)^4})\frac{1-e^{2n^2\varepsilon^2}}{n^2}|[\phi_n, x]|^2$$
$$>0.$$

Therefore (A^*, R) is approximately controllable.
Since $[T(t)x,\ t\geq 0]$ is represented as, for x in H

$$T(t)x=\sum_{n=-\infty}^{\infty}e^{-n^2 t}\phi_n[\phi_n, x],\quad t\geq 0$$

and $\lim_{n\to\pm\infty}e^{-n^2 t}=0$, $[T(t), t\geq 0]$ is compact. Next the resolvent $R(\lambda, A)$ becomes

$$R(\lambda,A)=\sum_{n=-\infty}^{\infty}\int_0^{\infty}e^{-\lambda t}e^{-n^2 t}\phi_n[\phi_n,x]dt,\quad \lambda>0$$
$$=\sum_{n=-\infty}^{\infty}\frac{1}{n^2+\lambda}\phi_n[\phi_n, x].$$

We have $\lim_{n\to\pm\infty}\left|\frac{1}{n^2+\lambda}\right|=0$, hence $R(\lambda,A)$ is also compact.
We now study stability of $[S(t),\ t\geq 0]$. Since

$$S(t)x=\sum_{n=-\infty}^{\infty}e^{[-n^2-\frac{1}{(n^2+1)^3}]t}\phi_n[\phi_n, x],$$

$$\|S(t)x\|\leq e^{-t}\|x\|$$

$[S(t),\ t\geq 0]$ is exponentially stable. Here we interprete stability of $[S(t),\ t\geq 0]$ by using the former results.

Remark 3 Since R is strictly positive, and the solution P of the SSRE exists, $[S(t),\ t\geq 0]$ is exponentially stable by Theorem 3.1.

Remark 4 The unstable supspace, span $\{\phi_0\}$ is controllable and observable, eventhough (A, B) is not exactly controllable and (R, A) is not exactly observable. Therefore $[S(t),\ t\geq 0]$ is exponentially stable.

Remark 5 The existance of P and (A, B) approximately controllability imply that $[S(t),\ T\geq 0]$ is weakly stable. On the other hand, since $[T(t),\ t\geq 0]$ is compact, so is $[S(t),\ t\geq 0]$, hence by Theorem 2.4, $[S(t),\ t\geq 0]$ is exponentially stable.

Remark 6 Even though R is strictly positive, (A^*,R) is not exactly controllable. This is an example where theorem 2.2 applies to this case where $[T(t),\ t\geq 0]$ is compact.

The above example explains many theorems discussed earlier. If we have a finite number of unstable states, we can stabilize the system (A, B) by compact feedback. However we cannot have exponential stability in the case where the original semigroup has an infinite number of unstable states as in Theorem 3.2.

4. Conclusions

We have studied stability of Co semigroup perturbed via the SSRE. In many practical applications, we use a finite number of sensors and actuators. Therefore the operator B and R is compact. If the original semigroup has an infinite number of unstable states, then we cannot have exponential stability. If the system has only a finite number of unstable states, then we can have exponential stability even though B is compact as in the example.

Most of the practical systems have inherent damping modes. These modes are stabilized with uniform decay rates. However very minute vibration or oscillation in the infinite dimensional systems which are to be persistent forever, cannot be stabilized with an exponential decay rate.

The next thing to exponential stability is strong stability and weak stability. These stabilization concepts must be considered to analyze these systems which are not exponentially stabilizable, for example, the system described by the wave equation.

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