Frequency Analysis on Parametric Resonance of Periodically Non-stationary Systems with Distributed Parameters

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ABSTRACT

This thesis presents methodology to evaluate the stability of such periodically non-stationary dynamic systems with distributed parameters in frequency domain and gives usability of the proposed method with an applied example.

2. Parametric resonance in periodically non-stationary systems with distributed parameters

As the systems with distributed parameters have theoretically infinite number of natural frequencies, parametric resonances in such systems occur in many regions of frequencies as well. Fig. (1) shows the characteristic of parametric resonances by combining parametric circle with frequency responses of systems with distributed parameters.

In case of the systems with distributed parameters it is more convenient to use central parametric circumference than current non-central parametric circumference owing to many cross points of frequency contours of the systems. When the periodically non-stationary system is not stable.
stationary element \( a(t) \) consists of mean value \( a_0 \) and periodically non-stationary signal \( a_i(t) \),

\[
a(t) = a_0 + a_i(t),
\]

(1)

\[
\frac{a_0}{a_0^2 - |a_i|^2}
\]

\[
W(j\omega) = -W_p^{-1}(j\varphi)
\]

Fig. 1. Condition of parametric resonances for periodically non-stationary systems with distributed parameters in complex plane.

Parametric circle can be positioned on the center of the complex plane by moving mean value of the signal to the non-periodical element. Then, the regions of instability can be easily found and experimentally obtained frequency responses can be also included.

In this case modified non-periodical transfer function becomes

\[
W_M(p) = \frac{W(p)}{1 + a_0 W(p)},
\]

(2)

Therefore, the first parametric resonance of the system can be found by simple relations;

\[
|W_M(j\frac{r\Omega}{2})| = |a_i| \quad \text{or} \quad |W_M(j\frac{r\Omega}{2})| = |a_i|^{-1}.
\]

(3)

Equation (3) can be graphically expressed in both complex and frequency-magnitude planes as shown in fig. 2. The advantage of the modified method is the possibility to use frequency-magnitude graph which is widely used by engineers and scientists in theory and practice. In many cases engineers can hardly find theoretical solutions of complex structures, but experimental results. The proposed method can easily include such experimental data for the analysis of non-stationary systems with distributed parameters.

On the point of transient response, general view of periodically non-stationary dynamic systems with distributed parameters can be depicted as shown in Fig. 3. For the computer evaluation of the non-stationary systems with distributed parameters it is necessary to model distributed parameters by well-known methods such as finite element method. Another way to evaluate time response of the system is to choose specific interesting mode from whole frequency responses and approximate the mode as the second order system.

![Graphical representation](image)

Fig. 2. Central parametric circumference with frequency response of systems with distributed parameters in complex plane (a) and frequency-magnitude plane (b).

![Graphical representation](image)

Fig. 3 Block diagram with frequency and time responses of the non-stationary systems with distributed parameters

### 3. Numerical example

M.I. Levinštein(6) performed an experimental research of parametric resonance on a synchronous generator with capacitance loading, but he considered a generator without motor. However, "motor-generator" system with long electrical line shall be considered in this thesis. Look at the "motor-generator" system as
shown in Fig. 4. In the frame of single-phased synchronous generator, it is not difficult to write the equation of motion with general variables;

\[ J\ddot{\alpha} = C_s I - M_c, U = R I + L \dot{I} + C_s \alpha, \]

\[ M_c = \frac{\partial M}{\partial \alpha} i_1, \frac{\partial M}{\partial t} i_1 = Z_s i_1, \]

(4)

Fig. 4 Motor-Generator electrical transformer system

Alteration of the mutual position between rotor and stator of the generator reflects that the inductance of the generator \( M \) can be considered to change by time. Ideal law of alteration \( M \) and \( \partial M/\partial t \) during constant shaft rotation is generally adapted\(^{(7)}\) as shown in Fig. 5.

Fig. 5. Alteration character of the mutual inductance according to time during shaft rotation

Smooth variation of this dependence is applied in this work.

\[ M(t) = C_U \sin \alpha = C_U \sin \omega_t t, \]

Applying Laplace transform to the system and arranging the equation by the relative angle of rotational shaft \( \alpha \), following nonlinear equation can be obtained;

\[ T_s T_u P^3 + T_s P^2 \alpha + p \alpha + k_1 W_s(p) \left( \frac{\partial M}{\partial \alpha} \right)^2 p \alpha \]

\[ + T_s k_1 W_s(p) \left( \frac{2}{\partial \alpha} \left( \frac{\partial^2 M}{\partial \alpha^2} \right) p^3 \alpha + \left( \frac{\partial M}{\partial \alpha} \right)^2 p^2 \alpha \right) = \frac{U}{C_e} \]

where \( k_1 = T_s T_u / J, T_s = L / R, T_u = R J / C_e C_M \).

As the mutual inductance \( M \) is the function of rotation angle \( \alpha \), equation (5) is nonlinear. Let us consider transient motion – rotation of the rotor with constant speed \( \omega_t \). For the purpose of stability research of the rotation equation in variational form is used.

Introducing small variation of angle \( \Delta \alpha \) from equal rotation \( \alpha = \omega_t t \), equation (5) can be written in view of

\[ T_s T_u p^3 \Delta \alpha + T_s P^2 \Delta \alpha + p \Delta \alpha + \left( \begin{array}{c} 2k_1 W_s(p) \left( \frac{\partial M}{\partial \alpha} \right)^2 \Delta \alpha \left( \frac{\partial M}{\partial \alpha} \right)^2 \Delta \alpha \right) \]

\[ + T_s k_1 W_s(p) \left( \frac{2}{\partial \alpha} \left( \frac{\partial^2 M}{\partial \alpha^2} \right) p^3 \alpha + \left( \frac{\partial M}{\partial \alpha} \right)^2 p^2 \alpha \right) \Delta \alpha = 0. \]

(6)

Induced harmonic stationarization\(^{(6)}\) of equation (6) gives following excitation condition of first parametric resonance.

\[ T_s T_u p^3 \left( T_u + \frac{1}{2} T_s q \right) p^3 + \left( 1 + \frac{1}{2} p \right) p \]

\[ + \left( \frac{1}{2} q T_u p^2 + \left( \frac{1}{2} q + p (2 T_r q) p + q \left( 1 + \frac{1}{2} p \right) \right) \right) = 0, \]

where \( q = k_1 W_s(p) C_u^2 \), and \( p = \omega_t / \Omega = j \Omega / 2 \), where \( \Omega \) - alternating frequency of parameter (in this research single pole machine \( \Omega = 2 \omega_t \)).

Equation (7) can be considered as characteristic polynomial with exponential multiplier of some linear stationary systems. Equality condition of left part is analogous to the searching condition on the boundary of system stability with feedback connection which gives following transfer function.

\[ W(j\Omega) = \frac{2 T_s q (j \Omega / 2)^2 + 1.5 q}{T_u T_s (j \Omega / 2)^2 + T_u q / 2 (j \Omega / 2)^2 + q / 2} \]

\[ W(j\varphi) = \left( \frac{-0.5 e^{-\varphi}}{j \Omega / 2} \right). \]

(9)

Hence, in accordance with Nyquist criterion, uniform rotation of the system (motor-generator) rotor with speed \( \omega_t \) can be unstable in condition of loading circuit \( W_s(p) \) and alternating mutual inductance \( M(t) = C_U \sin \omega_t t \) if Nyquist diagram of the closed-loop transfer function \( W(p) W_i(j \varphi) \) encloses point \((-1/j 0)\) on imaginary plane.

\[ W(j \Omega) W_i(j \varphi) = -1 \quad \text{or} \quad W(j \varphi) = -W^{-1}(2 \Omega) \]

Therefore, critical frequencies (and amplitudes) of changing parameter which causes first parametric resonance excitation can be found.

Let us consider electromechanical transformer on line (250m) with parameter: \( r_L = 1 \cdot 6 \) Ohm/m, \( l_e = 1 \cdot 5 \) H/m, \( c_L = 1 \cdot 8 \) F/m. Lump loading inductance \( L_s = 0.032 \) H is assumed to be attached at the end of line. Transfer function of the long line with reactive loading inductance can be written\(^{(9)}\) as

\[ W_s(p) = \frac{1}{(\sinh \xi / \gamma) \sinh \gamma N + \cosh \gamma N R(p)} \]

where \( R(p) = L_s p \) and \( N \) indicates the number of finite elements.

Nyquist diagram and amplitude-frequency characteristics of the given system with distributed long electrical line loading is shown in Fig. (6).
Calculation shows that the first parametric resonance can occur at following frequency outskirts: till 5.91; from 151.06 to 151.49; from 302.99 to 303.19; from 455.20 to 455.30 rad/sec. The most interesting resonance is the third one as it is closely located near the nominal frequency of rotation 300 rad/sec.

$$W(p) = \frac{W_2(p)}{1 + W_1(p)W_3(p)}$$

coincides with above equation (8).

In case that the transfer function of loading is hyperbolic it is not possible to use standard simulation tool of Simulink. However for the observation of first parametric resonance it is not always necessary to use exact transfer function of equation (11), but enough to use its approximation around interesting frequency range.

Experiment shows the presence of the first parametric resonance (see Fig. 8).

![Fig. 6 Conditions of first parametric resonance excitement on electromechanical transformer with long line (250m) loading.](image)

Fig. 6 Conditions of first parametric resonance excitement on electromechanical transformer with long line (250m) loading;

Standard commercial software Simulink (see Fig. 7) was used for the numerical experiment with equivalent transfer functions $W_1(p)$ and $W_3(p)$ as

$$W_1(p) = \frac{\frac{1}{2}k_cC_U\left(T_a p + 1\right)}{T_a T_M p^2 + T_M p + 1},$$

$$W_3(p) = \frac{\frac{1}{2}k_cC_U\left(ST_a p + 3\right)}{T_a T_M p^2 + T_M p + 1},$$

and $W_2(p) = W_a(p)$.

![Fig. 7. Block scheme of Simulink for numerical experiment.](image)

Fig. 7. Block scheme of Simulink for numerical experiment

It is easy to prove that in this case whole transfer function

![Fig. 8. Oscillogram of the vibration growth for the electromechanical transformer by numerical experiment.](image)

(a) general view, (b) fragment of (a) (dotted line – vibration of parameter, solid line – vibration in the system).

4. **Conclusion**

Frequency analysis of periodically non-stationary systems with distributed parameters was studied and found following basic results.

1) Famous method of «stationarization» of periodically non-stationary systems, which simplifies the search of parametric resonance excitement condition in frequency domain, is modified.

2) Essence of the modification comes from the separation of the stationary elements from the complex non-stationary parts and such procedure gives the central parametric circumference for the non-stationary element.

3) Calculation of electromechanical transformer system, which consists of constant current motor and
synchronous generator with distributed loading, was studied as an example of non-stationary system with distributed parameters.

4) Condition of loss stability of transformer rotation owing to the parametric resonance excitement was obtained.

References