BANACH ALGEBRA

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A normed linear space is a linear space N in which, to each vector x, there corresponds a real number, denoted by $\|x\|$ and called the norm of x, satisfying the following properties:

(1) $\|x\| \geqq 0$, and $\|x\| = 0$ if and only if $x=0$;

(2) $\|x+y\| \leqq \|x\| + \|y\|$;

(3) $\|\alpha x\| = |\alpha| \|x\|$.

The normed linear space N is a metric space with respect to the metric d defined by $d(x,y) = \|x-y\|$. A Banach space is a complete normed linear space.

Theorem 1. Let B(X,Y) be the set of all bounded linear operators of a normed space X into a normed space Y. Then B(X,Y) is complete if Y is complete, where B(X,Y) is the set of all bounded linear operators of X in Y.

Example 1. Let B(X) be the set of all bounded linear operators of a normed linear space X into itself. For A, $B \in B(X)$, $(AB)x = A(Bx)$. Then

$$\frac{\|ABx\|}{\|x\|} \leqq \frac{\|A\| \; \|Bx\|}{\|x\|} \leqq \frac{\|A\| \; \|B\| \; \|x\|}{\|x\|} = \|A\| \cdot \|B\|.$$

Hence $\|AB\| \leqq \|A\| \cdot \|B\|$.

B(X) is a normed space, and algebraically it is an algebra with the property $\|A\| \cdot \|B\| \geqq \|AB\|$.

Let **A** be a linear associative algebra with either the real or complex numbers as its field K of scalars. The algebra is called a normed algebra provided it is a normed space, satisfying the multiplicative inequality $\|xy\| \leqq \|x\| \cdot \|y\|$. If it is a Banach space, it is called a Banach algebra.

If a Banach algebra **A** has an identity e, then $\|e\| = \|ee\| \leqq \|e\| \cdot \|e\|$, so that $\|e\| \geqq 1$. Neverthless we may be able to renorm $\|e\| = 1$.

Example 2. By theorem 1, it is evident that B(X) is a Banach algebra provided that X is a Banach space.

Example 3. One of the most important Banach algebras, denoted by C(X), consist of all bounded continuous complex-valued functions defined on a topological space X.

Example 4. The sequences of complex numbers $a = \{a_n\}$ with $\|a\| = \sum_{-\infty}^{\infty} |a_n| < \infty$ and with multiplication $a_* b$ defined by $(a_* b)_n = \sum_{m=-\infty}^{\infty} a_{n-m} b_m$, is another Banach algebra.

Theorem 2. Every Banach division algebra is isomorphic to its scalar field.

Theorem 3. Let **A** be a commutative Banach algebra with multiplicative identity. Then any maximal ideal M, **A**/M is a division algebra. Accordingly, it is isomorphic to the scalar field

1. Involutions in Banach algebras

A Banach algebra **A** is called a Banach *-algebra if it has an involution, that is, if there exists a mapping $x \to x^*$ of **A** into itself with the following properties:

(1) $(x+y)^* = x^* + y^*$

(2) $(\alpha x)^* = \alpha x^*$

(3) $(xy)^* = y^* x^*$

(4) $(x^*)^* = x$.

It is an easy consequence of (4) that the involution $x \to x^*$ is actually a bijection of **A** onto itself. Furthermore, if (5) $\|x^* x\| = \|x\|^2$ is satisfied in a Banach *-algebra, it is called **A** B*-algebra.

Example 5. Let M be a maximal ideal of **A** with an identity. Then **A**/M is isomorphic to K. If K is a complex field, **A**/M is easily checked to be a B*-algebra. Let $\hat{x}(M)$ be the mapping:

$\hat{x}(M)=x(M)=x+M$. Then $\hat{x}(M)$ is a mapping of **M** into K where **M** is the set of all maximal ideals in **A**. If we put the $\hat{a}$-projective topology on **M**. **M** is called a maximal ideal space, where $\hat{a}=\{\hat{x}: x \in A\}$. The mapping $x \to \hat{x}$, of couse, we have identified $x+M$ with some scalar in K under the isomorphism, is called a Gelfand mapping. Then we have the following "Gelfand-Naimark Theorem":

Theorem 4. If **A** is a commutative B*-algebra, then the Gelfand mapping $x \to \hat{x}$ is an isometric *-isomorphism of **A** onto the commutative B*-algebra C(**M**). the set of all bounded continuous complex-valued functions on M.

If we apply this theorem to C(X), where X is a compact Hausdorff space, we have the following Banach-Stone theorem:

Theorem 5. Two compact Hausdorff spaces X and Y are homeomorphic if and only if their corresponding function algebras C(X), and C(Y) are isomorphic.

Historically speaking, a B*-algebra has originally been called a C*-algebra by Gelfand and Naimark, adding the following axiom: "$1+x^*x$ has an inverse". Later in the commutative case, Gelfand and Naimark proved in a rather intricate way that the last axiom is redundant. We note that $\| x \| = \| x^* \|$ is easily proved in the commutative case. A neat proof by Fukamiya is now available.

A commutative B*-algebra is simply the algebra of all continuous functions on a compact Hausdorff space with the *-operation complex conjugation (under isometric imbedding).

After a decade of mystery the noncommutative case of Gelfand and Naimak's query received its answer: the axiom "$1+x^*x$ has an inverse" can be omitted in the noncommutative case as well. The key lemma was discovered independently and nearly simultaneously by Fukamiya, and Kelley and Vaught. The Kelley-Vaught version is extremely brief and elegant and can be reproduced here. One has to show that, if x and y are positive elements in a B*-algebra, the same is true of $x+y$. Write $\| x \| = \alpha$, $\| y \| = \beta$. Then $\| \alpha - x \| \leq \alpha$, $\| \beta - y \| \leq \beta$, whence $\| (\alpha+\beta)-(x+y) \| \leq \alpha+\beta$.

2. Derivations

Two apparently unrelated results stimulated some recent research on derivations in Banach algebras.

In quantum mechanics one encounters unfounded operators satisfying $AB-BA=I$. Can this equation be satisfied with bounded operators? Wielandt proved that the answer is "No". Silov proved the following theorem. Let **A** be a Banach algebra of continuous functions on the unit interval. Suppose that **A** contains all n-fold differentiable functions. Then for some n, **A** contains all n-fold differentiable functions. We are at the moment concerned with the corollary that the algebra of all infinitely differentiable functions cannot be normed to form a Banach algebra.

The conjecture based on these two results is the following: if x, y are elements in a Banach algebra such that $xy-yx$ commutes with x, then $xy-yx$ is a generalized nilpotent. In the finite-dimensional case this is a well-known theorem of Jacobson. After a variety of partial results had been obtained, Kleinicke proved the conjecture. We introduce the inner derivation $a \to a' = ax - xa$. Our hypothesis states that $y''=0$. A simple induction based on Leibnitz's formula shows that $(y^n)^{(n)} = n!(y')^n$. If we write K for the norm of the bounded operator $a \to a'$, we then have $\| (y')^n \| \leq K^n \| y \|^n / n!$. It follows that y' is a generalized nilpotent element. From this result it is easy to proceed to the following theorem: Any continuous derivation of a commutative Banach algebra maps it onto its radical.

3. W*-algebra

We digress from the principal topic of W*-algebra to quote an example of a B*-algebra. We have already observed that the bounded linear operators of a Banach space into itself form a Banach algebra. In a Hilbert space H this normed algebra B(H) admits another important operation, the adjoint: that is, $(Tx, y) = (x, T^*y)$, because of "Riesz Representation Theorem" and definition of adjoint operators. Then the involution operation $T \to T^*$ has the following properties:

(1) $T^{**}=T$

(2) $(S+T)^*=S^*+T^*$

(3) $(\lambda T)^*=\lambda T^*$

(4) $(ST)^*=T^*S^*$

(5) $\| T^*T \| = \| T \|^2$

(6) $(I+T^*T) \in B(H)$, where I is the identity operator. Thus B(H) is a B*-algebra.

The typical neighborhood of 0 for the weak topology on B(H) is obtained by specifying a positive ε, a finite set of elements x_i, y_i in H and taking all $T \in B(H)$ with $|(Tx_i, y_i)| < \varepsilon$.

A W*-algebra is a weakly closed B*-algebra. Murray and Von Neumann made immense progress in the study of W*-algebras. An excellent summary of the work of Murray and Von Neumann is given by Naimark. The W*-algebra plays a vital role in studying group representations, especially infinite-dimensional representations.

4. Group algebras.

Let (X, S, μ) be a measure space. If $p \geq 1$, we shall denote by $L_p(X)$ the class of all measurable functions f for which $|f|^p$ is integrable with norm $\| f \|_p = (\int |f|^p d\mu)^{1/p}$.

$L_p(X)$ is a Banach space. Let G be a locally compact group. In every locally compact group G there exists at least one regular Haar measure.

$L^1(G)$ becomes a Banach algebra with multiplication $(f_* g)(x) = \int_G f(xy) g(y^{-1}) d\mu(y) = \int_G f(y) g(y^{-1}x) d\mu(y)$, which is called the convolution.

Theorem 6: $L^1(G)$ is commutative if and only if G is commutative.

Theorem 7: $L^1(G)$ has an identity if and only if G is discrete. This $L^1(G)$ is the algebra to which the expression "group algebra" is usually applied. The theory of commutative group algebras has been extremely well developed with the help of character groups. The treatise of Loomis in "Introduction to abstract harmonic analysis" gives a self-contained treatment of the algebra $L^1(G)$ for abelian G and the elementary theory for non-abelian G. Naimark's "Normed rings" and Hewitt and Ross's "Abstract Harmonic analysis I" contain a complete discussion of Banach algebra, including recent developments, as well as a text book treatment of those parts of functional analysis relevant to the theory of Banach algebra.

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