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PRIME IDEALS IN BANACH ALGEBRAS①

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INTRODUCTION

In his book [2] Rickart explained the representation theory of algebras and introduced the HULL-KERNEL topology on Π_{A} and Σ_{A} which are called the structure space and the strong structure space of the algebra A respectively. The former consists of all primitive ideals and the latter of all maximal modular 2-sided ideals in an algebra A.

In parallel with these, we introduce prime ideals according to McCoy [1] and the space Σ'_{A} the set of all proper prime ideals in algebra A.

The results given here to describe certain properties of prime ideals and establish some relations between primitive ideals and prime ideals in a (Banach) algebra A.

The highlight of CHAPTER I is to introduce notations and basic theorems which will be referred to CHAPTER II.

In CHAPTER II we propose to define prime ideals and Σ_{A} in an algebra A and to show the main result in this paper.

CHAPTER I. PRELIMINARIES

The purpose of this introductory chapter is to set up the notation and terminology to be used throughout and some basic known results. All unexplained notation and terminology will be found in [1] and [2].

Unless otherwise noted, the word "(Banach) algebra" means a real or complex associative (Banach) algebra throughout this paper.

Definition 1.1 Let A be any algebra over a field F. Let X be a linear space over the same field F and denote by L(X) the algebra of all linear transformations of X into itself.

Then any homomorphism of A into the algebra L(X) is called a **REPRESENTATION** of A on L(X) or on X. The representation is called FAITHFUL if the homomorphism is an isomorphism.

If F is either the reals or complexes and X is a normed linear space, then a homomorphism of A into the algebra B(X) of all bounded linear transformations of X into itself is called a **NORMED REPRESENTATION.**

For the case of a Banach algebra the term "representation" will, unless otherwise indicated, mean "normed representation".

The representation is said to be **STRICTLY IRREDUCIBLE** provided (0) and X are the only subspaces and is said to be **TOPOLOGICALLY IRREDUCIBLE** in the normed case provided these are the only closed invariant subspaces of X with respect to the range of the representation.

Among the representations of an algebra A there is the so-called **LEFT-REGULAR** representation on the linear space of A obtained by taking for each $a\in \mathsf{A}$ the linear transformatson T_a defined by $\mathrm{T}_a x=ax$ for $x\in\mathsf{A}$. It is obvious that the mapping $a\to\mathrm{T}_a$ is a representation of

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A.

Definition 1.2 Let L be a left ideal in an algebra A. Then we can consider the representation $a \to T_a^{A-L}$, which is induced on the difference space A-L by the left regular representation of A. This representatien is called **LEFT-REGULAR REPRESENTION** of A on A-L. The kernel of the representation consists of all $k \in A$ such that $kA \subseteq L$. This 2-sided ideal is called the QUOTIENT of the ideal L and is denoted by L:A.

Definition 1.3 A 2-sided ideal in an algebra A is called PRIMITIVE if it is the quotient of a maximal modular left ideal.

From the definition it is immediate that primitive ideals in A are necessarily proper.

Theorem 1.1 A 2-sided ideal of an algebra A is primitive if and only if it is the kernel of a strictly irreducible representation of A.

Theorem 1.2 In an algebra A,

(i) Any maximal modular 2-sided ideal is primitive.

(ii) Every modular 2-sided ideal is contained in a primitive ideal.

(iii) If P is primitive and L_1, L_2 are left ideals such that $L_1L_2 \subseteq P$, then either $L_1 \subseteq P$ or $L_2 \subseteq P$.

Theorem 1.3 If P is a 2-sided ideal in an algebra A, then the following conditions are equivalent;

(i) If B, C are 2-sided ideals in A such that $BC \subseteq P$, then $B \subseteq P$ or $C \subseteq P$.

(ii) If (a), (b) are principal ideals in A such that $(a)(b) \subseteq P$, then $a \subseteq P$ or $b \in P$.

(iii) If $a\,A\,b \subseteq P$, then $a \subseteq P$ or $b \subseteq P$.

(iv) If R_1, R_2 are right ideals in A such that $R_1R_2 \subseteq P$, then $R_1 \subseteq P$ or $R_2 \subseteq P$.

(v) If L_1, L_2 are left ideals in A such that $L_1L_2 \subseteq P$, then $L_1 \subseteq P$ or $L_2 \subseteq P$.

CHAPTER II. PRIME DEALS AND SOME RESULTS ON THEM

McCoy introduced prime ideals in general associative rings (See [1] or Theorem 1.3). This suggests that we are able to introduce this new concept in an algebra also. We to some extent develop some properties of prime ideals in the light of the theory of primitive ideals and maximal modular 2-sided ideals in algebras, in connection with the representation theory of A, and finally give some relations between prime ideals and primitive ideals in A. As a prelude to it we insert the so-called **HULL-KERNEL TOPOLOGY** on Σ_A, which is compatible, replacing primitive ideals or maximal modular 2-sided ideals by proper prime ideals in A (See [2]).

Definition 2.1. A 2-sided ideal in an algebra A is called PRIME if it satisfies any one (and therefore all) of the properties stated in Theorem 1.3.

Obviously A itself is a prime ideal in A but not proper.

A pirme ideal of A is called **MAXIMAL PRIME** if it is different from A and is not properly contained in any prime ideal other than A.

From this definition and Theorem 1.2 we have

Theorem 2.1 In an algebra A,

(i) Any primitive ideal is prime.

(ii) Any maximal modular 2-sided ideal is prime.

(iii) Every modular 2-sided ideal is contained in a proper prime ideal.

Proof. (i) This statement is a consequence of Theorem 1.2 (iii) and Theorem 1.3(v).

(ii) By Theorem 1.2(i) any maximal modular 2-sided ideal is primitive and hence it is prime by(i).

(iii) By Theorem 1.2(ii) every modular 2-sided ideal is contained in a primitive ideal. Since primitive ideal is a proper prime ideal in A, this implies that every modular 2-sided ideal is contained in a proper prime ideal, and hence the theorem is proved.

Theorem 2.2 Let A be an algebra with an identity element e. Then any maximal prime ideal in A is primitive.

Proof. Let P be any maximal prime ideal in A. Since $e \in A$, P is a modular 2-sided ideal in A. Then by Theorem 1. 2(ii) there exists a maximal modular left ideal L such that $P \subseteq L:A \subseteq L \subsetneqq A$ Since the primitive ideal L:A is proper prime and P is maximal prime, it follows that P= L:A. This proves theorem.

We shall now make the following;

Definition 2.2 An algebra A is said to be PRIME in case the zero ideal is a prime ideal.

We have already stated that any primitive ideal is prime. Thus the primitive algebra in which the zero ideal is primitive is always a prime algebra. The following result goes to the opposite direction of this.

Theorem 2.3 A prime algebra A which contains a minimal left ideal is a primitive algebra.

Proof. Let L′ be a minimal left ideal in a prime algebra A and Ae(L′) a left annihilator of L′ in A. Then Ae(L′) is a 2-sided ideal in A such that Ae(L′)L′=10. Since A is prime, this implies that Ae(L′)=(0). Now we define a linear transformation $T_a{}^{L'}$ on L′ as $T_a{}^{L'}l=al$ for given $a \in A$ and $l \in L'$. Then $a \longrightarrow T_a{}^{L'}$ for $a \in A$ is a faithful strictly irreducible representation of A on L′. By Theorem 1.1, the zero ideal(0) which is the kernel of this representation of A is primitive. This implies that A is primitive, and the proof is completed.

Now we consider the space Σ_A of all proper prime ideals in an algebra A with the **HULL-KERNEL TOPOLOGY**. The injection mapping of Σ_A and Π_A into Σ_A is obviously a homeomorphism.

Lemma. Let I be a modular 2-sided ideal in an algebra A. Then the hull of I in Σ_A is compact.

Proof. Let $\{F_\lambda\}$ be any family of closed subsets of $h[I]$ with finite intersection property. Denote K the smallest 2-sided ideal of A which contains all of the ideals. $k(F_\lambda)$. Suppose $k=A$. and e is an identity modulo I. Then $e \in A=K$. Therefore there exist $m_i \in k(F_{\lambda i})(i=1,2,\cdots,n)$ such that $e=m_1+m_2+\cdots\cdots+m_n$. Since $I \subseteq k(F_\lambda)$ for all λ, we see that $k(F_{\lambda 1})+\cdots\cdots+k(F_{\lambda n})=A$. This means $F_{\lambda 1}\cap\cdots\cap F_{\lambda n}=\phi$, contrary to the hypothesis. Therefore $K \neq A$. Since $F_\lambda \subseteq h(I)$, we have $I \subseteq k(h(I))=k(F_\lambda)\subseteq K$ and modularity of I implies modularity of K. Consequently, by Theorem 2. (iii), there exists $P \in \Sigma_A$ such that $K \subseteq P$. Since F_λ is closed in $h(I)$ and $h(I)$ is closed in Σ_A, $P \in h(P) \subseteq h(k(F_\lambda))=F_\lambda$ for every λ. Hence $P \in \cap_\lambda F_\lambda \neq \phi$, which proves that $h(I)$ is compact.

Theorem 2.4 If an algebra A has an identity element e, then Σ_A is compact.

Proof. Since the zero ideal (0) is a modular 2-sided ideal in A, $h((0))=\Sigma_A$ is compact by Lemma, and hence theorem is proved.

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