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A NOTE ON TENSOR PRODUCT OF TOPOLOGICAL LINEAR SPACES

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INTRODUCTION

The main purpose of the present paper is to describe several properties of the tensor product of two topological linear spaces.

Suppose that E and F are locally convex spaces. Then E⊗F with the projective tensor product topology (cf. Chap. 2 for definiton) is a Hausdorff locally convex space. If further E and F are metrizable barrelled spaces, then so is E⊗F.

E⊗F can be given the topology of bi-equicontinuous convergence, which is weaker than the projective tensor product topology. With this topology, E⊗F is also a Hausdorff space.

In Chap. 1 we list some basic notation and preliminary results concerning algebraic tensor product of linear spaces and some properties of a linear topological space, which will be needed for the later development.

In Chap. 2, 3 we describe some properties of tensor product of two topological linear spaces.

CHAP. 1 NOTATIONS AND PRELIMINARIES

E's and F's are linear spaces over the same scalar field K.

Then the following is a well known result: Let $E=\Sigma_i E_i$ (direct) and $F=\Sigma_j F_j$ (direct). Then the cannonical mapping of $E_i \otimes F_j$ into E⊗F is an isomorphism, for each (i, j). Furthermore if we identify $E_i \otimes F_j$ with its image under the above cannonical isomorphism, then we have

$$E\otimes F=\sum_{i,j} E_i \otimes E_j \text{ (direct)}.$$

Suppose that (ξ_λ) and (η_μ) are bases of E and F respectively. Then $E=\sum_\lambda K\xi_\lambda$ (direct) and $F=\sum_\mu K\eta_\mu$ (direct), whence we get

$$E\otimes F=\sum_{\lambda,\mu} K(\xi_\lambda \otimes \eta_\mu) \text{ (direct)}.$$

Thus $(\xi_\lambda \otimes \eta_\mu)$ forms a basis for E⊗F.

Suppose that A and B are non-void convex subsets of a linear topological space E and that the interior of A is non-void. Then Separation Theorem asserts that there is a continuous linear functional on E separating A and B iff B is disjoint from the interior of A.

Suppose that E is a locally convex linear topological space, and that A and B are non-void disjoint convex subsets of E. Then there is a continuous linear functional strongly separating A and B iff O is not a member of the closure of B-A.

Let $(E, \mathscr{T})$ be a locally convex linear topological space.

If A is convex and $\mathscr{T}$-closed in E, and if x is an element which is not in A, then there exists a $\mathscr{T}$-continuous linear functional f strongly separating A and $\{x\}$ since x-A, $\mathscr{T}$-closed set in E, does not contain zero.

Since f is weakly continuous, no net in A can converge weakly to x, whence A is $w(E, E^*)$-closed.

Since $\mathscr{T}$ is stronger than $w(E, E^*)$-topology, a convex subset of a locally convex space $(E, \mathscr{T})$ is $\mathscr{T}$-closed iff it is $w(E, E^*)$-closed.

CHAP. 2 PROJECTIVE TOPOLOGICAL TENSOR PRODUCT

Suppose that E and E are linear spaces with algebraic duals E′ and F′ respectively. For $x \in E$ and $y \in F$, let's define $x \otimes y$ by $x \otimes y(x', y') = \langle x, x' \rangle \langle y, y' \rangle$ for every $(x', y') \in E' \times F'$. Then clearly $x \otimes y$ is a bilinear form on $E' \times F'$.

Let $E \otimes F$ be the linear span of $k(E \times F)$ where k is the mapping of $E \times F$ into $B(E', F')$, the space of all bilinear forms on $E' \times F'$, and $k(x, y) = x \otimes y$.

PROPOSITION 2. 1.

The space of all linear forms on $E \otimes F$ is isomorphic to the space of all bilinear forms $E \times F$.

Proof. Define π: $(E \otimes F) \to B(E, F)$ by $\pi(f) = f \cdot k$ for all $f \in (E \otimes F)'$. Clearly it suffices to show that π is bijective. Let $g \in B(E, F)$. Define f: $E \otimes F \to K$ by $f(x \otimes y) = g(x, y)$ and

$f(\sum_i x_i \otimes y_i) = \sum_i g(x_i, y_i)$. Suppose that $\sum_{i=1}^{n} x_i \otimes y_i = 0$ and that $x_i = \sum_\lambda a_{i\lambda} \xi_\lambda$ and $y_i = \sum_\mu b_{i\mu} \eta_\mu$ where (ξ_λ) and (η_μ) are bases for E and F respectively. Then $\sum_{i=1}^{n} x_i \otimes y_i = \sum_{i=1}^{n} \sum_{\lambda,\mu} a_{i\lambda}\ b_{i\mu} \xi_\lambda \otimes \eta_\mu = 0$.

Since $(\xi_\lambda \otimes \eta_\mu)$ is a basis for $E \otimes F$, $\sum_{i=1}^{n} a_{i\lambda}\ b_{i\mu} = 0$ for each (λ, η). On the other hand

$$\sum_{i=1}^{n} g(x_i, y_i) = \sum_{i=1}^{n} \Big[\sum_{\lambda,\mu} a_{i\lambda}\ b_{i\mu} g(\xi_\lambda, \eta_\mu) \Big]$$

$$= \sum_{\lambda,\mu} (\sum_{i=1}^{n} a_{i\lambda}\ b_{i\mu}) g(\xi_\lambda, \eta_\mu) = 0$$

Thus the mapping f is well defined. By the definition of f, $g = f \circ k = \pi(f)$. If $f \circ k = 0$, then $f = 0$ on $k(E \times F)$ and so is on $E \otimes F$. Thus the assertion follows.

PROPOSITION 2. 2. If G is a third linear space, the space of all linear mappings of $E \otimes F$ into G is isomorphic to the space of all bilinear mappings of $E \times F$ into G.

Proof. Let's designate by $L(E \otimes F; G)$ and $L(E, F; G)$ the space of all linear mappings of $E \otimes F$ to G and the space of all bilinear mappings of $E \times F$ into G respectively.

Define $\pi: L(E, F; G) \to L(E \otimes F; G)$ by $\pi(f)(x \otimes y) = f(x, y)$. Then the remaining proofs are quite the same with the previous ones.

Suppose hereforth in this section that E and F are locally convex topological linear spaces.

PROPOSITION 2. 3. There is one and only one locally convex topology for $E \otimes F$ such that, for every locally convex space G, the space of all continuous linear mappings of $E \otimes F$ into G corresponds to the space of all continuous bilinear mappings of $E \times F$ into G, that is, the isomorphism in (Prop. 2. 2.) preserves continuity.

Proof. Let $\mathscr{U}$ and $\mathscr{V}$ be local bases for E and F respectively, and let $\mathscr{T}$ be the topology for $E \otimes F$ having as a local base the convex circled extensions of the sets $U \otimes V = \{x \otimes y : x \in U, y \in V\}$ as U and V run through $\mathscr{U}$ and $\mathscr{V}$ respectively.

Then the following statesments are immediate:

1) each element of $\mathscr{T}$ is convex, circled and radial at 0,

(2) $U_1 \otimes V_1 \cap U_2 \otimes V_2 \supset (U_1 \cap U_2) \otimes (V_1 \cap V_2)$,

(3) for all non-zero scalar a, the convex circled extension of $a(U \otimes V)$ is again an element of $\mathscr{T}$.

Thus $\mathscr{T}$ is a locally convex vector topology for $E \otimes F$.

Suppose that f is a continuous bilinear mapping of $E \times F$ into a third locally convex space G. Then there exist U and V neighborhoods of 0 in E and F resp. such that $f(U, V) \subset W$ where W is a neighborhood of 0 in G. Then $\pi(f)(U \otimes V) = f(U, V) \subset W$. This shows the continuity of $\pi(f)$. The converse is also immediate.

Let $\mathscr{T}'$ be another locally convex topology for $E \otimes F$ winth the above property.

Taking $G = (E \otimes F, \mathscr{T}')$, we get that $\mathscr{T}'$ is finer than $\mathscr{T}$.

Again exchanging the role of $\mathscr{T}$ and $\mathscr{T}'$, we get the uniqueness of such a topology.

The algebraic tensor product $E \otimes F$, equipped with the above topology is called the "projective tensor product" and will be denoted by $E \otimes_1 F$.

Considering the above proposition, we get the following immediately.

PROPOSITION 2. 4. The topology of $E\otimes_1 F$ is the strongest locally convex topology for which the cannonical bilinear mapping of $E\times F$ onto $E\otimes_1 F$ is continuous.

PROPOSITION 2. 5. If E and F are Hausdorff spaces, then so is $E\otimes_1 F$.

Proof. Let z^* be a non-zero element of $E\otimes_1 F$. Then $z^*=\sum_{i=1}^{n} x_i\otimes y_i$ for $x_i\in E$ and $y_i\in F$. Without loss of generality we may assume that $\{x_i\}$ and $\{y_i\}$ are linear independent sets in E and F resp.

Let V be a circed neighborhood of 0 in F such that $y_1 \notin V$. If y_1 belongs to the closure of $s[y_i]_{2\leq i\leq n}$ the span of $\{y_i\}$ $(i=2, \cdots, n)$, then there exist scalars a_i's such that $\sum_{i=2}^{n} a_i y_i$ belongs to y_1+V. This contradicts to the linear independence of the set $\{y_i\}$. Hence 0 does not belong to the closure of $y_1-s[y_i]_{2\leq i\leq n}$. Then by Separation Theorem there exists a continuous linear functional g strongly separating $\{y_1\}$ and $s\{y_i\}$, that is, $|g(y_1)|>\sup\{|g(y)|\}$ $y\in s[y_i]$

We may assume that $g(y_1)>1$ and then $g(y_i)=0 (2\leq i\leq n)$.

Similarly we can select a continuous linear functional f on E such that $f(x_1)>1$.

If we set $U=\{x\in E: |f(x)|\leq 1\}$ and $V=\{y\in F: |g(y)|\leq 1\}$, then $|z(f, g)|\leq 1$ for all $z\in U\otimes V$ and $|z^*(f, g)|>1$.

Let's denote by $E\otimes_1 F$ the completion of $E\otimes F$ with the projective tensor product topology.

PROPOSITION 2. 6. If E and F are metrizable, then $E\hat{\otimes}_1 F$ is a Fréchet space. If E and F are metrizable barrelled spaces, then so is $E\otimes_1 F$.

Proof. Since E and F are metrizable, there exist $\mathscr{U}$ and $\mathscr{V}$ countable local bases for E and F resp. Then clearly $\{U\otimes V: U\in\mathscr{U}, V\in\mathscr{V}\}$ is countable, whence $E\otimes_1 F$ is metrizable. Thus its completion is a Fréchet space.

Let's show the latter assertion.

A locally convex space E is a barrelled space iff each $w(E^*, E)$-bounded subset of the adjoint E^* is equi-continuous. By (PROPOSITION 2. 3) the space of all continuous bilinear forms on $E\times F$ is isomorphic to the space of all continuous linear forms on $E\otimes_1 F$.

Hence it suffices to show that every pointwise bounded family of continuous bilinear forms on $E\times F$ is equi-continuous.

Let $\{f_i: i\in I\}$ be the family. For each closed convex circled neighborhood W of 0 in the scalars and for each element x of E, $W_{xi}=\{y: f_i(x, y)\in W\}$ is a barrel in F for any i of I. Whence $\{y; f_i(x, y)\in W, \forall i\in I\}$ is closed, convex and circled, Since $\{f_i: i\in I\}$ is pointwise bounded, given $(x, y)\in E\times F$ there exists $k_{xy}>0$ such that $\{f_i: i\in I\}\subset k_{xy}\{f: |f(x, y)|\leq 1\}$. Let y be any element of F and let r be a positive real number such that $r\, k_{xy}\in W$.

Then for each element i of I, $|f_i(x, ry)| = r|f_i(x, y)|\leq rk_{xy}$.

Since W is circled, this means that $\{y; f_i(x, y)\in W \forall i\in I\}$ is radial at 0. Thus $\{y: f_i(x, y)\in W\ \forall i\in I\}$ is a barrel in F, whence if $y_n\to 0$, $\{f_i(x, y_n): i\in I, n=1, 2, \cdots\}$ is bounded. Therefre $\{x: f_i(x, y_n)\in W, \forall i\in I, n=1, 2, \cdots\}$ is radial at 0, and it follows that it is barrel in E.

If $x_n\to 0$, $\{f_i(x_n, y_n): i\in I, n=1, 2, \cdots\}$ is bounded. Thus the assertion follows.

CHAP. 3 TOPOLOGICAL TENSOR PRODUCT OF BI-EQUICONTINUOUS CONVERGENCE

Suppose that E and F are locally convex Hausdorff spaces with adjoints E^* and F^* resp.

For each $x\in E$ and $y\in F$, the bilinear functional $x\otimes y$ on $E'\times F'$ defines by restriction a bilinear functional on $E^*\times F^*$, which is separately continuous when E^* and F^* have their w*-topology.

On the space of all separately continuous bilinear functionals on $E^* \times F^*$, the topology of uniform convergence on products of equi-continuous subsets of E^* and E^* is a vector topology.

The relative topology for $E \otimes F$ is called the "topology of bi-equicontinuous convergence" and the tensor product space equipped with this topology will be denoted by $E \otimes_2 F$.

PROPOSITION 3. 1. The topology of bi-equicontinuous convergence is the topology of uniform convergence on the sets $U^\circ \otimes V^\circ$, as U and V run through local bases for E and F resp.

Proof. Let W be a neighborhood of 0 in $E \otimes_2 F$. Then there exist A and B equicontinuous subsets of E^* and F^* resp. such that $W = \{x \otimes y : |<x, A> <y, B>| \leq 1\}$. Since A and B are equicontinuous, there exist U and V elements of local bases for E and F resp. such that $A \subset U^\circ$ and $B \subset V^\circ$. Hence $A \otimes B \subset U^\circ \otimes V^\circ$.

Thus W is a neighborhood of 0 in the topology of uniform convergence on the sets $U^\circ \otimes V^\circ$.

Conversely for all U and V elements of local bases for E and F respectively, U° and V° are equi-continuous in E^* and F^* respectively so that the assertion follows.

PROPOSITION 3. 2. $E \otimes_2 F$ is a Hausdorff space.

Proof. Let $x^* \otimes y^*$ be a non-zero element of $E \otimes_2 F$. Then there exist U and V, closed convex neighborhoods of 0 in E and F respectively such that $x^* \notin U$ and $y^* \notin V$.
Let $W = \{x \otimes y \in E \otimes_2 F : |<x, U^\circ> <y, V^\circ>| \leq 1\}$.
Then $x^* \otimes y^* \notin W$.
If otherwise, $|<x^*, U^\circ>| \leq 1$ or $|<y^*, V^\circ| \leq 1$, say $|<x^*, U^\circ>| \leq 1$. Then $x \in U^\circ$, which is identical with U since E is locally convex.

PROPOSITION 3. 3. The topology of bi-equicontinuous convergence is weaker than the projective tensor product topology.

Proof. If A and B are equicontinuous subsets in E^* and F^* resp., then there exist U and V elements of local bases for E and F resp. such that $A \subset U^\circ$ and $B \subset V^\circ$.
Hence $A_\circ \supset U^\circ{}_\circ$ and $B_\circ \supset V^\circ{}_\circ$. Since E and F are locally convex, we may assume that U and V are closed and convex.

In this situation we have already found that $U^\circ{}_\circ = U$ and $V^\circ{}_\circ = V$. Thus $A_\circ \supset U$ and $B_\circ \supset V$, so that $A_\circ \otimes B_\circ \supset U \otimes V$. Now the assertion follows.

BIBLIOGRAPHY

(1) J.L. Kelley and I. Namioka, *Linear Topolngical Spaces*, D. Van Nostrand Co., 1963.

(2) A. E. Taylor, *Introduction to Functional Analysis*, Wiley Inc., 1963.

(3) C. Chevalley, *Fundamental Concepts of Algebra*, Academic Press, 1956.

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