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A NOTE ON THE BOCHNER-HERGLOTZ-WEIL-RAIKOV THEOREM

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INTRODUCTION

The well known theorem of Bochner-Herglotz-Weil-Raikov states that a linear functional f defined on a semi-simple, self-adjoint, commutative Banach algebra A is positve and extendable if and only if there exists a finite positive Baire measure u on the maximal ideal space of A such that $f(x)=\int \hat{x}\, du$ for every x in A.

It is the purpose of this paper to seek for conditions that may be replaced to the semisimplicity and selfadjointness of the assumption for A.

A modified form of the Bochner-Herglotz-Weil-Raikov Theorem is stated in Theorem 2.1 and Theorem 2.2 below.

In case when A is a complex algebra (not necessarily a Banach algebra), the above conditions may be replaced by the assumption that A have an involution and a positive functional f on A.

In case when A is a Banach algebra the above conditions may be replaced by the assumption that A has a continuous involution.

By the same argument as in pp. 98—99 of [1], it then comes out that the Plancherel theorem [1, 26J] can also be modified in a similar fashion as will be remarked at the end of this paper.

The highlight of CHAPTER I is to introduce notations and basic theorems which will be referred to CHAPTER II.

CHAPTER I. PRELIMINARIES

The purpose of this introductory chapter is to set notation and terminology and to list some basic known results that will be of later use. All unexplained notations and terminologies will be found in [1] and [2]. Unless otherwise noted, the word "(Banach) algebra" means a complex (Banach) algebra throughout this paper.

Definition 1.1 A mapping $x \to x^*$ defined on an algebra A is an involution if it has at least the first four of the following properties.

1. $x^{**}=x$
2. $(x+y)^*=x^*+y^*$
3. $(\lambda x)^*=\bar{\lambda}x^*$ (λ is a complex scalar)
3. $(xy)^*=y^*x^*$
5. $\|xx^*\|=\|x\|^2$
6. $-xx^*$ has a quasi-inverse ($e+xx^*$ has an inverse)

By a *-algebra, we shall mean an algebra having an involution. A *-algebra is said to be symmetric if it further satisfies (6). A Banach algebra with an involution satisfying all the properties (1) to (6) is said to be a C*-algebra.

Definition 1.2 We denote the space of all regular maximal ideals of an algebra A by M. If $m=m^*$ for any m in M, then m is said to be a symmetric regular maximal ideal of A. The radical of a commutative algebra is the intersection of its all regular maximal ideals. If the radical is zero, the algebra is said to be semi-simple.

Definition 1.3 Let Δ be the space of all continuous non-zero homomorphisms of an algebra A onto the complex numbers. For every x in A, the function $\hat{x}$ on Δ is defined by $x(h)=h(x)$,

$h \in \Delta$. We denote by $\hat{A}$ the algebra of all such funtions $\hat{x}$ on Δ. A commutative Banach algebra A is said to be self-adjoint if for every x in A there exists a unique y in A such that $\hat{y}=\bar{\hat{x}}$.

Let A be any algebra over a field F and X be a linear space over the same field F. We denote by L(X) the algebra of all linear transformations of X into itself, by B(X) the algebra of all bounded linear transformations of X into itself, and by C(X) the alhebra of all complex-valued continuous functions on X.

Any homomorphism of A into the algebra L(X) is called a representation of A on X. Among the representations of an alhebra A, there is the so-called left (right) regular representation on the linear space A obtained by taking for each a in A the linear transformation T_a defined by $T_a x = ax$ ($T_a x = xa$) for x in A. The representation T_x is a *-representation if L(X) has an involution and it is true that $T_x{}^* = (T_x)^*$.

Definition 1.4 If A is a *-algebra, a linear functional f on A is said to be positive if $f(xx^*) \geq 0$ for all x in A, and a positive functional f is called extendable if it satisfies the conditions $|f(x)|^2 \leq k f(xx^*)$ and $f(x^*)=\overline{f(x)}$.

For a fixed positive functional f defined on a dense ideal A_0 of an (Banach) algebra A, an element $p \in A_0$ is said to be positive definite if the functional f_p defined on A by $f_p(x)=f(px)$ is positive and extendable.

For convenience, we state here theorems of Bochner-Herglotz-Weil-Raikov and Plancherel, for the proof of which the reader is referred to pp. 97—99 of [1].

Theorem 1.1 (Herglotz-Bochner-Weil-Raikov). If A is a semi-simple, self-adjoint, commutative Banch algebra, then a linear functional f on A is postitive and extendable if and only if there exists a finite positive Baire measure u on M such that $f(x)=\int \hat{x}\,du$ for every x in A.

Theorem 1.2 (Plancherel) Let A be a semi-simple, self-adjoint, commutative Banach algebra, and let f be a positive functional defined on dense ideal A_0 in A. Then there is a unique Baire measure u on M such that $\hat{p} \in L'(u)$ and $f(px) = \int (\hat{x}\hat{p})\,du$ whenever p is positive definite with respect to f.

For the proofs of the following theorems which will be used in CHAPTER II, see p. 214, p. 226 and p. 229 of [2].

Theorem 1.3 Every positive functional f in Banach algebra with a continuous involution satisfies the inequality $f(xaa^*x^*) \leq K a f(xx^*)$, where Ka is a constant determined by a.

Theorem 1.4 In any Banach algebra A having a continuous involution and a positive and extendable functional f on A, there exists a pseudo-norm $|x|$ for which $|f(x)| \leq K \cdot |x|$.

Theorem 1.5 The pseudo-norm $|x|$ in A, given by Theorem 1.4, is equal to max $|x(m)|$ for $m \in M_0$, where M_0, is the space of all symmetric regular maximal ideals of A.

CHAPTER II. BOCHNER-HERGLOTZ-WEIL-RAIKOV THECREM ON A COMPLEX ALGEBRA AND BANACH ALGEBRA

Theorem 1.1 and Theorem 1.2 have assumed that A is a semi-simple, self-adjoint commutative Banach algebra, that is, have assumed that $\hat{x}^* = \bar{\hat{x}}$. But the hypothesis that $\hat{x}^* = \bar{\hat{x}}$ can be dropped both of Theorem 1.1 and Theorem 1.2. We shall start with any commutative *-algebra and Banach *-algebra.

Lemma 2.1 Let f be a positive functional on a commutative *-algebra A. Then $K=\{x : f(xx^*)=0,\ x \in A\}$ is a right ideal of A and the quotient space A/K is a pre-Hilbert space.

Proof. Let x, z be in K and y be in A. Then by Cauchy-Bunyakovsky's inequality $|f(xy^*)|^2 \leq f(xx^*) \cdot f(yy^*) = 0$, $f((x+z)\cdot(x+z)^*) = f(xx^*) + f(xz^*) + f(zx^*) + f(zz^*) = 0$, and $f(xaa^*x^*) =$

$f(x(xaa^*)^*)=0$ for all a in A. Thus K is a right ideal of of A.

We denote residue classes with respect to K by $\bar{x}$, $\bar{y}$, $\bar{z}$, ……with representative x, y, z, ……, respectively, and define $(\bar{x}, \bar{y})=f(xy^*)$ for arbitrary x and y of A.

To see that $(\bar{x}, \bar{y})$ is well defined, observe that

$$f((x+k_1)(y+k_2)^*)=f(xy^*)$$
$$+f(k_1y^*)+f(xk_2^*)+f(k_1k_2^*).$$

If k_1 and k_2 are in K, then $f(k_1y^*)=f(xk_2^*)=0$, and $(\bar{x}, \bar{y})$ is well defined.

The function $(\bar{x}, \bar{y})$ with $\bar{x}$ and $\bar{y}$ ranging over the quotient space A/K has usual properties of an inner product. Specifically, since $f(xy^*)=\overline{f(yx^*)}$, $(\bar{x}, \bar{y})=f(xy^*)=\overline{f(yx^*)}=\overline{(\bar{y}, \bar{x})}$. $(\bar{y}, \bar{x})$ is a linear function for fixed $\bar{y}$ and taken into its complex conjugate by interchanging of $\bar{x}$ and $\bar{y}$. Since $f(x)$ is a positive functional, we have $(\bar{x}, \bar{x})=f(xx^*)\geq 0$. If $(\bar{x}, \bar{x})=0$, then $f(xx^*)=0$, which shows that x belongs to K, *i. e.*, $\bar{x}=0$.

This implies that K is a right ideal of A and the quotient space A/K is a pre-Hilbert space. This completes the proof.

We denote by $a\to U_a$ the right regular representation of A on A/K.

Lemma 2.2 Let A be a commutive *-algebra (Banach algebra having a continuous involution) and f be a positive functional satisfying the condition $f(xaa^*x^*)\leq Kaf(xx^*)$ (f be a positive functional). Then for any a in A U_a is bounded with respect to the f inner product norm and $a\to U_a$ is a *-represention of A on the Hilbert completion of A/K.

Proof. Let $\|\bar{x}\|$ be the f-inner product norm in A/K. By Lemma 2.1 and Theorem 1.3, for any a in A and $\bar{x}$ in A/K $\|U_a\bar{x}\|^2=(U_a\bar{x}, U_a\bar{x})=f(xaa^*x^*)\leq Kaf(xx^*)=Ka(\bar{x}, \bar{x})=Ka\|\bar{x}\|^2$.

Thus U_a is bounded and in thia case U_a can be uniquely extended to the Hilbert space which is the completion of A/K.

Morever,

$$U_{a+b}\bar{x}=\overline{x(a+b)}=xa+xb=U_a\bar{x}+U_b\bar{x},$$
$$U_{ab}\bar{x}=\overline{x(ab)}=U_b\overline{(xa)}=U_aU_b\bar{x},$$
$$(U_a\bar{x}, \bar{y})=f(xay^*)=f(x(ya^*)^*)=(\bar{x}, U_a{}^*\bar{y}).$$

This implies that $a\to U_a$ is a *-representation of A. This completes the proof.

The set $B=\{U_a: a\in A\}$ is a C *-algebra under the invloution $U_a\to U_a{}^*$. [1, p. 28] We denote the space of all complex valued non-zero homomorphisms on B by Δ_1.

Lemma 2.3 Let h' be an element of Δ_1. Define h be a functional on A such that $h(a)=h'(U_a)$. Then h is an element of Δ and $M_f=\{h: h'\in\Delta_1, h(a)=h'(U_a)\}$ is a closed subset of Δ in the weak topology.

Proof. Let h' be an element of Δ_1 and h be a functional such that $h(a)=h'(U_a)$. Since $U: a\to U_a$ is a *-representation of A, $h'U$ is a homomorphism of A into the complex numbers and $h(a)=h'(U_a)=h'(U(a))=h'U(a)$, h is an element of Δ.

We designate $h'U$ by U^*h'. If $h'\neq h''$, then $U^*h'\neq U^*h''$. Thus U^* is a 1-1 mapping of Δ_1 into Δ.

The toplogy of Δ_1 is the weak topology defined by the algebra of functions $(\widehat{U(a)})$. But $\widehat{U(a)}(h')=h'(U(a))=(U^*h')(a)=\hat{a}(U^*h')$ and since the function $\hat{a}$ define the topology of Δ, the mapping U^* is a homeomorphism.

Now let h be any homomorphism of Δ in the closure of $U^*(\Delta_1)$, that is, for x_1, ……, x_n, there exists $h'\in\Delta_1$ such that $|h(x_i)-h'(U(x_i))|<\varepsilon$, $i=1$, ……, n. This implies that, if $U(x_1)=U(x_2)$, then $h(x_1)=h(x_2)$ so that the function h' defined by $h'(U(x))=h(x)$ is single valued on B. Thus h' is a homomorphism of B onto complex numbers and $h(x)=h'(U(x))$, *i. e.*, $h=U^*h'$. Hence $U^*(\Delta_1)$, is closed in Δ. This completes the proof.

Theorem 2.1 Let A be a commutative *-algebra with a positive functional f on A such that $f(xaa^*x^*)\leq Kaf(xx^*)$. Then a linear functional F on A is positive, extendable and $|F(x)|\leq K\|\hat{x}\|_1$

if and only if there exists a finite positive Baire measure u_f on M_f such that $F(x)=\int \hat{x} du_f$ for x every x in A, where $\|\hat{x}\|_1$ is the uniform norm of $\hat{x}$ restricted on M_f.

Proof. By above Lemmas, for any x in A and h in M_f $\overline{\hat{x}(h)}=\hat{x}^*(h)$. If $F(x)=\int \hat{x} du_f$ where u_f is a finite positive Baire measure on M_f, then

$$F(xx^*)=\int |\hat{x}|^2 du_f \geq 0,$$

$$F(x^*)=\int \hat{x}\ du_f=(\int \bar{\hat{x}}\ du_f)^-=\overline{F(x)},$$

$$|F(x)|^2=\left|\int \bar{\hat{x}}\ du_f\right|^2 \leq (\int |\bar{\hat{x}}|^2\ du_f)(\int 1\ du_f) =K\ F(xx^*),$$

$$|F(x)|=\left|\int \hat{x}\ du_f\right| \leq \int \|\hat{x}\|_1\ ud_f=\|\hat{x}\|_1 \int 1\ du_f =k\|\bar{\hat{x}}\|_1;$$

that is, F is positive and extendable. Conversely, if F is positive, extendable and $|F(x)|\leq K\|\hat{x}\|_1$, then th linear functional I_F defined on $\hat{A}_f=\{\hat{x}: h(x)=\hat{x}(h),\ h\in M_f\}$ by $I_F(\hat{x})=F(x)$ is bounded and since $\hat{A}_f$ is dense in $C(M_f)$ by Stone-Weierstrass theorem, I_F can be extended to $C(M_f)$.

If $f'\in C(M_f)$ and $f'\geq 0$, then $f'^{\frac{1}{2}}$ can be uniformly approximated by functions $\hat{x}\in A_f$ and f' can be uniformly approximated by functions $|\hat{x}|^2$. Since $I_F(|x|^2)=F(xx^*)\geq 0$ and $I_F(|\hat{x}|^2)$ approximates $I_F(f')$, it follow that $I_F(f')\geq 0$. That is, I_F is a bounded intergral. If u_f is related measure, we have the desired result $F(x)=\int \hat{x}\ du_f$ for all x in A.

Theorem 2.2. Let A be a commutative Banach algebra having a continuous involution. Then a linear functional f on A is positive and extendable if and only if there exists a finite positive and extendable if and only if there exists a finite positive Baire measure u on M_0 such that $f(x)=\int x du$ for every x in A, where $\hat{x}$ is restricted on M_0 which is the space of all symmetric regular maximal ideals of A.

Proof. Since M_0 is the space of all regular symmetric maximal ideals of A, for any x in A and h in M_0 $\overline{\hat{x}(h)}=\hat{x}^*(h)$. If $f(x)=\int \hat{x}\ du$ where u is a finite positive Baire measure on M_0, then

$$f(xx^*)=\int |\hat{x}|^2\ du \geq 0,$$

$$f(x^*)=\int \hat{x}^*\ du=(\int \hat{x}\ du)^-=\overline{f(x)},$$

$$|f(x)|^2=\left|\int \hat{x}\ du\right|^2 \leq (\int |\hat{x}|^2\ du)(\int 1\ du) =Kf(xx^*);$$

that is, f is positive and extendable. Conversely, if f is positive and extendable, then by Theorems 1.4 and 1.5 $|f(x)|\leq K|x|=K\max|\hat{x}(m)|$ for $m\in M_0$. Therefere f may be regarded as a bounded linear functional on $\hat{A}_f=\{\hat{x}: m\in M_0\}$. Since $\hat{A}_f$ is dense in $C(M_0)$ by Stone-Weierstrass theorem, the bounded linear functional I_f defined on A_f by $I_f(\hat{x})=f(x)$ can be extended in a unique way to $C(M_0)$.

If $f'\in C(M_0)$ and $f'\geq 0$, then $f'^{\frac{1}{2}}$ can be uniformly approximated by functions $\hat{x}$. Since $I_f(|\hat{x}|^2)=f(xx^*)\geq 0$ and $I_f(|\hat{x}|^2)$ approximates $I_f(f')$, it follows that $I_f(f')\geq 0$. That is, I_f is a bounded integral. If u is related measure, we have desired result $f(x)=\int \hat{x}\ du$ for all x in A.

Remarks 1. Theorem 1.2 can be extended to the *-algebra having the same assumptions as in Therem 2.1.

2. Therem 1.2 can be extended to the Banach *-algebra having the same assumptions as in Theorem 2.2.

BIBLIOGRAPHY

[1] Lynn H. Loomis, *An introduction to abstract harmonic analysis*, New York, 1953.

[2] Charles E. Rickart *General theory of Banach algebras*, New York, 1960.

[3] M. A. Naimark, *Normed rings*, The Netherlands, 1959.

[4] Irving Kaplansky, *Topological algebras*, Chicago, 1958.

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