SELECTIONS AND UNITARY ACTIONS OF SEMIGROUPS.

By J. T. Borrego and E. E. DeVun

1. Introduction. An action is a continuous function α: $T\times X\to X$ where T is a (topological) semigroup, X is a Hausdorff space, and $\alpha(t_1t_2, x)=\alpha(t_1, \alpha(t_2, x))$. We shall also assume that T and X are compact and that α is onto. We write tx for $\alpha(t, x)$ and AB for $\{tx|t\in A, x\in B\}$. The action α induces a closed quasi-order $\{(x,y)|Tx\subset Ty\}$ on X [2] ; $M(\alpha)$ is the set of all maximal elements of X under this quasi-order. The α-orbit of a point x in X is Tx. An action is called unitary if $x\in Tx$ for all $x\in X$. Here we shall be concerned with unitary actions. The reader is referred to [4], [5], [9], and [10] for information concerning the general theory of semigroups.

A multi-valued function F from X to Y associates with each $x\in X$ a non empty subset $F(x)$ of Y. F is continuous if and only if $\{x_n\}$ is a net convergence to x implies $F(x_n)$ converges to $F(x)$ [8] and $F(x)$ is closed for all $x\in X$. Associated with any action α there are two multivalued functions $F: T\to X$ defined by $F(t)=t(M(\alpha))^*$ (where * indicates topological closure) and $G: X\to X$ defined by $G(X)=Tx$. These functions are continuous [9]. Here we are interested in the converse, i. e., given F and G when is it possible to construct a unitary action α such that $\alpha(T\times\{x\})=G(x)$ for all $x\in X$. We shall give conditions on F and G which enable us to construct a disjoint unitary action of T on X. Using this construction we shall give a new proof of a theorem due to Stadtlander [6]. The methods used here are very similar to those of [2] and [8].

The reader is referred to [7] for a more complete treatment of multivalued functions.

2. Main Theorem. An *aw*-homomorphism between two actions $\alpha_1: T_1\times X_1\to X_1$ and $\alpha_2: T_2\times X_2\to X_2$ is a pair (g,f) where g is a continuous homomorphism of T_1 onto T_2, f is a continuous function of X_1 onto X_2 and $f\alpha_1(t,x)=\alpha_2(g(t), f(x))$ for all $t\in T$ and all $x\in X$. An action α is *disjoint* if and only if $\{Tx|x\in M(\alpha)\}$ is pairwise disjoint. The following proposition enables us to restrict our attention to disjoint actions.

PROPOSITION 1. *If $\alpha:T\times X\to X$ is a unitary action, then there is a compact, Hausdorff space Y and an action $\beta:T\times Y\to Y$ such that β is disjoint, $M(\beta)$ is closed and α is an aw-homomorphic image of β.*

This proposition is very similar to Theorem 5 of [2] and a slight modification of the proof to that theorem will prove this proposition.

Let X be a compact Hausdorff space and K be a continuous multivalued function of X onto X. Then $P(K)=\{(x,y)\mid K(x)\subset K(y)\}$ is a closed quasi-order on X and let $M(K)$ be the set of maximal elements of X under $P(K)$. A disjoint unitary orbit function on X is a continuous multivalued function G of X onto X such that $x\in G(x)$ for all $x\in X$, if $x\in G(y)$ then $G(x)$ is contained in $G(y)$ and $\{G(b)\mid b\in M(G)\}$ is a pairwise disjoint collection of subsets of X. Let T be a compact semigroup. A T-selector for a disjoint unitary orbit function G on X is a continuous multivalued function $F:T\to X$ such that for $b\in M(G)$, the function $f_b:T\to G(b)$ defined by $f_b(t)=F(t)\cap G(b)$ is a left-multiplicative single-valued onto function and if $x\in F(t)\cap G(b)$, $b\in M(G)$, then $G(b)\cap F(Tt)=G(x)$. (A left-multiplicative function h on a semigroup T is a function such that $\{(t,t')\mid h(t)=h(t')\}$ is a left congruence of T.)

The following remark indicates the motivation for the above definition.

REMARK 2. Let $\alpha:T\times X\to X$ be a disjoint, unitary action. Then $G:X\to X$ defined by $G(x)=Tx$ is a disjoint, unitary orbit function. If $B=B^*\subset M(\alpha)$, $TB=X$ and card $G(x)\cap B=1$ for $x\in M(\alpha)$, then $F:T\to X$ defined by $F(t)=tB$ is a T-selection.

The proof is routine.

THEOREM 3. *Let X be a compact Hausdorff space, G be a disjoint unitary orbit function on X, T be a compact semigroup, and let F be a T-selector for G. If $\alpha:T\times X\to X$ is defined by $\alpha(t,x)=f_b(tf_b^{-1}(x))$ where $b\in M(G)$ and $x\in G(b)$, then α is a disjoint unitary action with $\alpha(T\times\{x\})=G(x)$.*

PROOF. Since f_b is left multiplicative for $b\in M(G)$ and $\{G(b)\mid b\in M(G)\}$ is pairwise disjoint, α is well-defined.

Next, we shall show that α is continuous. Let $\{t_n\}$ be a net in T converging to t, $\{x_n\}$ be a net in X converging to x, $b_n\in M(G)$ such that $x_n\in G(b_n)$. Let $b\in M(G)$ such that $x\in G(b)$, let $t_n'\in f_{b_n}^{-1}(x_n)$, z be a cluster point of $\{\alpha(t_n, x_n)\}$ and let t' be a cluster point of $\{t_n'\}$. By selecting subnets we may suppose $\{\alpha(t_n, x_n)\}$ converges to z and $\{t_n'\}$ converges to t'. Since $F(Tt_n')\cap G(b_n)=G(x_n)$, $\alpha(t_n, x_n)$

$=f_{b_n}(t_n f_{b_n}^{-1}(x_n))\in G(x_n)$ and thus $z\in G(x)\subset G(b)$. Because $\alpha(t_n, x_n)\in F(t_n t_n')$, $z\in F(tt')\cap G(b)=f_b(tt')$. But $x_n\in F(t_n')$ implies $x\in F(t')$ so that $\alpha(t,x)=f_b(tt')=z$.

It is easily shown that $\alpha(t_1,\alpha(t_2,x))=\alpha(t_1t_2,x)$ for $t_1,t_2\in T$ and $x\in X$ and that $\alpha(T\times\{x\})=G(x)$ for $x\in X$.

A K-space is a pair (X,K) where X is a compact metric space and K is a continuous multivalued function from X onto X such that:

(1) If $x\in K(y)$, then $K(x)\subset K(y)$

(2) If $K(x)=K(y)$, then $x=y$

(3) $x\in K(x)$ for all $x\in X$

(4) $K(x)$ is a metric arc (homeomorphic to $[0,1]$ or a point) with one endpoint x and one endpoint in $L(K)=\{x\in X\mid x \text{ is minimal in } P(K)\}$.

(5) Card $(K(x)\cap L(K))=1$ for all $x\in X$.

This definition is different in form to the definition given by Stadtlander [6] but the two definitions are equivalent.

A thread is a semigroup which is homeomorphic to $[0,1]$ and in which one endpoint is an identity and the other is a zero. The following corollary concerning thread actions can be found in [6]. The proof presented there is different.

COROLLARY 4. *Let T be a thread and (X,K) be a K-space. Then there is a unitary action T on X with $0X=L(K)$ where 0 is the zero of T.*

PROOF. Define $p:X\to L(K)$ by $p(x)=L(K)\cap K(x)$. Then p is a retraction [9].

Carruth [3] has shown that there is a metric d for X which is convex with respect to $P(K)$, i.e., $K(x)\subset K(y)\subset K(z)$ implies $d(x,y)\le d(x,z)$. We may also assume d is bounded by 1 and $T=[0,1]$. Thus, the function $k:M(K)^*\to T$ by $k(b)=d(b,p(b))$ is continuous.

Let $Y=\bigcup\{K(b)\times\{b\}\mid b\in M(K)^*\}$. Define $G:Y\to Y$ by $G(y,b)=K(y)\times\{b\}$. It is easily verified that G is a disjoint unitary orbit function and $M(G)=\{(b,b)\mid b\in M(K)^*\}$. Define $F:T\to Y$ by $F(t)=\{(x,b)\mid d(x,p(b))=tk(b)\}$. By an argument similar to the one used in Theorem 2.6 of [9], F is continuous and it is routine to verify that F is a T-selector for G. Let α be the action given by Theorem 1. Let $\pi_1:Y\to X$ be the first projection. It is a simple computation to verify that if $\pi_1(y)=\pi_1(x)$ then $\pi_1\alpha(t,x)=\pi_1\alpha(t,y)$ for $t\in T$. Thus, there is an action β from $T\times X$ onto X defined by $\beta(t,x)=\pi_1\alpha(t,y)$ where $\pi_1(y)=x$ [1, 2].

University of Massachusetts, Amherst, Massachusetts
Wichita State University, Wichita, Kansas

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