LOCAL DISCRETE EXTENSIONS OF TOPOLOGIES

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In this paper, we introduce a new concept of extension called as local discrete extensions. This concept is motivated by [2]. Let $(X, \mathscr{T})$ be a topological space and A be a subset of X. Then the topology $\mathscr{T}[A]=\{U\sim B \mid U\in\mathscr{T}, B\subset A\}$ is called a local discrete extensions of $\mathscr{T}$ by A. It is clear that $\mathscr{T}[A]$ is a topology for X, for $\bigcup_\alpha(U_\alpha\sim B_\alpha)=\bigcup_\alpha U_\alpha\sim(A\sim\bigcup_\alpha(U_\alpha\sim B_\alpha))$ for some $(U_\alpha\sim B_\alpha)\in\mathscr{T}[A]$. A topological space is called $\aleph$-compact if each open cover has a subcover whose cardinal is less than or equals to $\aleph$. This $\aleph$ denote an arbitrary cardinal number.

We attempt to investigate that, if $(X, \mathscr{T})$ has some topological property P, under what conditions will $(X, \mathscr{T}[A])$ also have property P. For a subset A of a topological space X, $\mathrm{cl}\,A$ denotes $\mathscr{T}$-closure of A, Int A denotes $\mathscr{T}$-interior of A, and $\mathrm{cl}^*\,A$ denotes $\mathscr{T}[A]$-closure of A, and $\mathrm{Int}^*\,A$ denotes $\mathscr{T}[A]$-interior of A, and B' denotes the complement $X\sim B$ of B. The termiology coincides with Kelley [1].

LEMMA 1. *Let A be any subset of $(X, \mathscr{T})$. Then $(A, \mathscr{T}[A]\cap A)$ is a discrete space.*

PROOF. It is clear from the definition of $\mathscr{T}[A]$.

THEOREM 2. *Let A be a closed subset of $(X, \mathscr{T})$. Then $(A, \mathscr{T}\cap A)$ is a discrete subspace of $(X, \mathscr{T})$ if and only if $\mathscr{T}=\mathscr{T}[A]$.*

PROOF. Let $U\sim B$ be any open in $(X, \mathscr{T}[A])$. Since A is a closed in $(X, \mathscr{T})$, B is a closed in $(X, \mathscr{T})$. Therefore $U\sim B$ is an open in $(X, \mathscr{T})$. Hence we have $\mathscr{T}=\mathscr{T}[A]$. The converse follows directely from lemma 1.

THEOREM 3. *Let $(X, \mathscr{T})$ be a topological space and $\mathscr{T}[A]$ be a local discrete extension of $\mathscr{T}$. Let B be any subset of $(X, \mathscr{T})$. Then*

(1) $\mathrm{cl}^*B=(A\cap B)\cup\mathrm{cl}(A'\cap B)$.

(2) $\mathrm{Int}^*B=(A'\cup B)\cap\mathrm{Int}(A\cup B)$. *In Particular, if A is a closed subset of $(X, \mathscr{T})$, then* $\mathrm{cl}^*\,(A\cup B)=\mathrm{cl}(A\cup B)$ *and* $\mathrm{Int}^*\,(A\cup B)=\mathrm{Int}(A\cup B)$.

PROOF : (1). If $A\subset B$, then $\mathrm{Int}^*\,B=\mathrm{Int}B$. If A and B are disjoint, then $\mathrm{cl}^*B=\mathrm{cl}B$. Therefore we have $\mathrm{cl}^*\,B=\mathrm{cl}^*(A\cap B)\cup\mathrm{cl}^*(A'\cap B)$. Since $A\cap B$ is a $\mathscr{T}[A]$-

closed, $\mathrm{cl}^*B=(A\cap B)\cup\mathrm{cl}(A'\cap B)$. (2) follows immediately from (1).

THEOREM 4. *If $(X,\mathscr{T})$ is regular or normal and A is an open subset of X, then $(X,\mathscr{T}[A])$ is regular or normal.*

PROOF. We prove the theorem only for regular. Let A be an open subset of $(X,\mathscr{T})$. Then every subset of A is $\mathscr{T}[A]$- open set. Let F be a closed subset of $(X, \mathscr{T}[A])$ and let $x\notin F$. Then there exists a $\mathscr{T}[A]$-open set $U\sim B$ such that $F=(U\sim B)'$. Hence $x\notin U'$ and $x\notin B$. There are two cases. Case (i) $x\notin A$. Since $(X,\mathscr{T})$ is regular, for each $x\notin U'$, there exist disjoint open sets U and V such that $x\in U$ and $U'\subset V$. Hence there are disjoint $\mathscr{T}[A]$- open sets $U\sim A$ and $V\cup B$ such that $x\in U\sim A$ and $F\subset V\cup B$. Case (ii). $x\in A$. this is clear.

THEOREM 5. *If $(X, \mathscr{T})$ is completely regular and A is an open subset of X, then $(X,\mathscr{T}[A])$ is completely regular.*

PROOF. Let V be a $\mathscr{T}[A]$- open and let $x\in V$. Then there exists a $\mathscr{T}[A]$-open set $U\sim B$ such that $V=(U\sim B)'$. Since $(X,\mathscr{T})$ is completely regular, there is a $\mathscr{T}$- continuous function f on X to $[0,1]$ such that $f(x)=0$ and f is identically one on $X\sim U$. Defining $f^*(x)=\begin{cases} f(x) & \text{on } (U\cap B)' \\ 1 & \text{on } U\cap B, \end{cases}$

then f^* is a $\mathscr{T}[A]$-continuous function on X to $[0,1]$. For, there are two cases. Case (i). $y\notin A$. Since f is a $\mathscr{T}$- continuous, there is a $\mathscr{T}$- neighourhood $N(y)$ of y such that $f(N(y))\subset N(f(y))$. Therefore $f^*(N(y)\sim A)\subset N(f(y))$, and hence f^* is a $\mathscr{T}[A]$- continuous. Case (ii). $y\in A$. Since $\{y\}$ is a $\mathscr{T}$-open, it is clear.

COROLLARY 6. *If $(X,\mathscr{T})$ is Tychonoff and A is an open subset of X, then $(X, \mathscr{T}[A])$ is Tychonoff.*

REMARK 7. In case that A is not an open subset of $(X,\mathscr{T})$, in general. above theorem 4. 5. does not hold.

EXAMPLE (1). Let $X=\{a,b,c\}$ and $\mathscr{T}=\{\phi,X\}$. Then $(X,\mathscr{T})$ is regular and normal. But $(X,\mathscr{T}[\{a,b\}])$ is neither.

EXAMPLE (2). Let $X=\{a,b\}$ and $\mathscr{T}=\{\phi,X\}$. Then $(X,\mathscr{T})$ is completely regular. But $(X,\mathscr{T}[a])$ is not.

THEOREM 8. *If $(X,\mathscr{T})$ is a second countable space, then $(X,\mathscr{T}[A])$ is a second countable space if and only if A is a countable subset of $(X,\mathscr{T})$.*

PROOF. "only if". Let $\mathscr{L}$ be a countable base of $(X,\mathscr{T})$ and let $\mathcal{L}_A=\{B\sim A_\alpha | B\in\mathcal{L}, A_\alpha$ is cofinite subset of $A\}$. Then, $\mathcal{L}_A$ is a countable and a base of $\mathcal{T}[A]$. For, let $x\in B\sim A_1$ and $A_1\subset A$. If A_1 is a cofinite subset in A, it is clear. If A_1 is not cofinite subset in A, then $A\sim A_1$ is not finite. There are two cases. Case (i). $x\notin A$. Then $x\notin B\sim A\subset B\sim A_1$. Case (ii). $x\in A$. Then $x\in B\sim(A\sim\{x\})\subset B\sim A_1$.

"If". Suppose that A is not a countable. Since $(A, \mathcal{T}[A]\cap A)$ is a discrete subspace by lemma 1. and A is not countable, $(A, \mathcal{T}[A]\cap A)$ is not second countable space. It is a contradiction.

THEOREM 9. *Let A be any subset of $(X,\mathcal{T})$. Then $(X,\mathcal{T})$ is a first countable if and only if $(X,\mathcal{T}[A])$ is a first countable space.*

PROOF. *Let* $\{U_i | i=1, 2, \cdots\cdots\}$ be a countable local base at any point x of X. There are two cases. Case (i). $x\notin A$. Then $\{U_i\sim A | i=1, 2, \cdots\cdots\}$ is a countable local base of a point x of $(X,\mathcal{T}[A])$. Case (ii). $x\in A$. Then $\{U_i\sim(A\sim\{x\}) | i=1, 2, \cdots\cdots\}$ is a countable local base of a point x of $(X,\mathcal{T}[A])$.

THEOREM 10. *Let $(X,\mathcal{T})$ be a $\aleph$-compact (countably compact). Then A has the cardinal number $\aleph$(finite) if and only if $(X,\mathcal{T}[A])$ is a $\aleph$-compact (countably compact).*

PROOF. We prove the theorem only for the $\aleph$-compact. "only if". Let $\{U_\alpha\sim B_\alpha | \alpha\in A\}$ be an open covering of $(X,\mathcal{T}[A])$. Then $\{U_\alpha | \alpha\in\mathfrak{A}\}$ is an open covering of $(X,\mathcal{T})$. Since $(X,\mathcal{T})$ is a $\aleph$-compact, there is a subcovering $\{U_\beta | \beta\in\mathcal{L}, \mathcal{L}\subset\mathfrak{A}\}$ of $\{U_\alpha | \alpha\in\mathfrak{A}\}$, where $|\mathcal{L}|\leq\aleph$. Choose $U_r\sim B_r$ such that $a\in U_r\sim B_r$ for each $a\in\mathfrak{A}$. Let $\mathcal{S}=\{r\in\mathfrak{A} | a\in U_r\sim B_r$ for each $a\in A\}$. Since $|A|=\aleph$ $|\mathcal{S}|\leq\aleph$. Hence we have $X=A\cup(X\sim A)=[\cup\{U_r\sim B_r | r\in\mathcal{S}\}]\cup[\cup\{U_\beta\sim B_\beta | \beta\in\mathcal{L}\}]=\cup\{U_\delta\sim B_\delta | \delta\in\mathcal{L}\cup\mathcal{S}\}$, where $\mathcal{L}\cup\mathcal{S}\subset\mathfrak{A}$, $|\mathcal{L}\cup\mathcal{S}|\leq\aleph$. Hence $(X, \mathcal{T}[A])$ is a $\aleph$-compact. "If". Suppose that $|A|\neq\aleph$. Since A is a $\mathcal{T}[A]$-closed, $(A,\mathcal{T}[A]\cap A)$ is $\aleph$-compact. On the other hand, $(A,\mathcal{T}[A]\cap A)$ is a discrete subspace by lemma 1. Since $|A|\neq\aleph$, $(A,\mathcal{T}[A]\cap A)$ is not a $\aleph$-compact. It is a contradiction.

THEOREM 11. *If A and B are subsets of $(X,\mathcal{T})$, then $\mathcal{T}[A][B]=\mathcal{T}[A\cup B]$.*

PROOF. This is clear from the definition of $\mathcal{T}[A]$.

THEOREM 12. *Let $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ be topological spaces. If A and B are subsets* *of X and Y respectively, then* $(\mathscr{T}\times\mathscr{U})\ [A\times B]\ \subset\mathscr{T}[A]\times\mathscr{U}[B]$.

PROOF. Let $U\in(\mathscr{T}\times\mathscr{U})\ [A\times B]$. Then $U=G\sim C$, where $G\in\mathscr{T}\times\mathscr{U}$ and $C\subset A\times B$. If $x\in U$, then there is a basic open set $E\times F$ such that $x\in E\times F\subset G$. Let $x=(x_1, x_2)$. Then we have a basic open set $E\sim(A\sim\{x_1\})\times F\sim(B\sim\{x_2\})$ in $\mathscr{T}[A]\times\mathscr{U}[B]$ such that $x\in E\sim(A\sim\{x_1\})\times F\sim(B\sim\{x_2\})\subset G\sim C$. Hence $G\sim C\in\mathscr{T}[A]\times\mathscr{U}[B]$.

REMARK 13. The converse inclusion of theorem 12 need not be true. For example, let X be the real line with usual topology ξ. Let A be the set of all rational numbers. Then $\xi[A]\times\xi[A]$ is not contained in $(\xi\times\xi)\ [A\times A]$.

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