ON PAIRWISE CONNECTED BITOPOLOGICAL SPACES

By Ivan L. Reilly

If X is a set and $\mathscr{T}_1$ and $\mathscr{T}_2$ are topologies on X, then Kelly [2] called the triple $(X, \mathscr{T}_1, \mathscr{T}_2)$ a bitopological space. Pervin [3] initiated the study of connectedness properties for such spaces, and Birsan [1] has also discussed this topic. This paper proves some further results in this area and, in particular, deals with total disconnectedness for bitopological spaces.

Following Pervin [3] we have the

DEFINITION 1. $(X, \mathscr{T}_1, \mathscr{T}_2)$ is ***pairwise connected*** iff X cannot be expressed as the union of two non-empty disjoint sets A and B such that $(A \cap \mathscr{T}_1 \text{ cl} B) \cup (\mathscr{T}_2 \text{ cl} A \cap B) = \phi$. (Throughout this paper $\mathscr{T}_1$ cl A denotes the $\mathscr{T}_1$ closure of the set A.) If X can be so expressed we write $X = A|B$ and this is a separation of X, which is then pairwise disconnected.

That the pairwise connectedness of $(X, \mathscr{T}_1, \mathscr{T}_2)$ is not governed by the connectedness of the topological spaces $(X, \mathscr{T}_1)$ and $(X. \mathscr{T}_2)$ is shown by the following examples. Let $X = \{a, b\}$, $\mathscr{T}_1$ be the discrete topology and $\mathscr{T}_2$ the indiscrete topology for X. Then $(X, \mathscr{T}_1, \mathscr{T}_2)$ is parwise connected while $(X, \mathscr{T}_1)$ is not connected. Let $\mathscr{T}_3$ be $\{\phi, X, \{a\}\}$ and $\mathscr{T}_4$ be $\{\phi, X, \{b\}\}$. Then $X = \{a\} | \{b\}$ is a separation of $(X, \mathscr{T}_3, \mathscr{T}_4)$, but $(X, \mathscr{T}_3)$ and $(X, \mathscr{T}_4)$ are connected.

DEFINITION 2. A function $f:(X, \mathscr{T}_1, \mathscr{T}_2) \to (Y, \mathscr{S}_1, \mathscr{S}_2)$ is ***parwise continuous*** (respectively, ***a pairwise homeomorphism***) iff the induced functions $f:(X, \mathscr{T}_1) \to (Y, \mathscr{S}_1)$ and $f:(X, \mathscr{T}_2) \to (Y, \mathscr{S}_2)$ are continuous (respectively, homeomorphisms).

Amongst other results, Pervin [3] proves the following.

THEOREM 1. *$(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise connected iff every pairwise continuous function $f:(X, \mathscr{T}_1, \mathscr{T}_2) \to (R, \mathscr{L}, \mathscr{R})$ has the Darboux property, that is, its range is* *an interval, where R is the set of real numbers and $\mathscr{L}$ and $\mathscr{R}$ are the left hand and right hand topologies on R with bases $\{(-\infty, x) : x \in R\}$ and $\{(y, +\infty) : y \in R\}$ respectively.*

DEFINITION 3. (Pervin). A subset K of $(X, \mathscr{T}_1, \mathscr{T}_2)$ is *pairwise connected* if the bitopological space $(K, \mathscr{T}_1 | K, \mathscr{T}_2 | K)$ is pairwise connected.

In contrast to the topological situation, closures of pairwise connected sets need not be pairwise connected. For example, if X is {a, b, c}, $\mathscr{T}_1$ is $\{\phi, X, \{a\}, \{b, c\}\}$ and $\mathscr{T}_2$ is $\{\phi, X, \{a\}, \{b\}, \{a, b\}\}$, then $\mathscr{T}_2$ cl $\{a\} = \{a, c\}$ and is pairwise disconnected.

THEOREM 2. *If $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise T_1, then every non-empty non-singleton pairwise connected subset of X is infinite.*

PROOF. Let A be a finite non-empty non-singleton subset of X. As $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise T_1 each singleton subset of X is $\mathscr{T}_1$ closed and $\mathscr{T}_2$ closed. Thus if B is any non-empty proper subset of A, B is $\mathscr{T}_1$ closed (as B is finite) and A$\sim$B is $\mathscr{T}_2$ closed. Hence B is $\mathscr{T}_1 | A$ closed and $\mathscr{T}_2 | A$ open, and so $(A, \mathscr{T}_1 | A, \mathscr{T}_2 | A)$ is pairwise disconnected, by theorem $A(c)$ of Pervin [3].

This result can be sharpened for the special case of quasi-metric spaces.

THEOREM 3. *If $(X, \mathscr{T}_1, \mathscr{T}_2)$ is a quasi-metric bitopological space, then every non-empty non-singleton pairwise connected subset of* X *is uncountable.*

PROOF. Let $\mathscr{T}_1$ be generated by the quasi-metric p on X, so that $\mathscr{T}_2$ is generated by the conjugate q of p. Let A be a non-empty non-singleton pairwise connected subset of X, and $a \in A$. Define the real valued function f on A by $f(x) = p(x, a)$ for each $x \in A$. Then f is $\mathscr{T}_1$ lower and $\mathscr{T}_2$ upper semi-continuous, so that the function $f : (A, \mathscr{T}_1 | A, \mathscr{T}_2 | A) \rightarrow (R, \mathscr{L}, \mathscr{R})$ is pairwise continuous. Thus, by theorem 1, $f(A)$ is an interval. Now for any point b of A distinct from a, $p(b, a)$ is non-zero. Thus the interval $(0, p(b, a))$ is contained in $f(A)$ which is therefore uncountable, and hence so is A.

The number of components of a bitopological space and the structure of each component is a bitopological invariant.

THEOREM 4. *Let $f:(X, \mathscr{T}_1, \mathscr{T}_2)\to(Y, \mathscr{S}_1, \mathscr{S}_2)$ be pairwise continuous. Then the image of each component of X lies in a component of Y. Furthermore, if f is a pairwise homeomorphism, then f induces a one to one correspondence between the components of X and those of Y, corresponding components being pairwise homeomorphic.*

PROOF. Let x be any point of X and C be the component of X containing x. Then $f(C)$ is pairwise connected, by Theorem D of Pervin [3], and contains $f(x)$. Hence $f(C)$ is contained in the component D of Y containing $f(x)$. If f is a pairwise homeomorphism, so is f^{-1}, so that $f(C)\subset D$ and $f^{-1}(D)\subset C$. Thus $f(C)=D$. The proof of the rest of the theorem is immediate.

DEFINITION 4. $(X, \mathscr{T}_1, \mathscr{T}_2)$ is *pairwise totally disconnected* if each pair of points of X can be separated by a separation of X, that is, if x and y are distinct points of X there is a separation $X=A|B$ such that $x\in A$ and $y\in B$ or $x\in B$ and $y\in A$.

The next result shows that this definition coincides with that of Birsan [1].

PROPORSITION 1. *The components of a pairwise totally disconnected space are its points.*

PROOF. Simply observe that any subset containing more than one point is pairwise disconnected.

If X is any non-empty non-singleton set, $\mathscr{T}_1$ is the discrete topology and $\mathscr{T}_2$ the indiscrete topology on X, then $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise connected, while $(X, \mathscr{T}_1)$ is totally disconnected.

If $X=\{a,b,c,d\}$, $\mathscr{T}_1$ is $\{\phi, X, \{b\}, \{a,b\}, \{b,d\}, \{a,b,d\}\}$ and $\mathscr{T}_2$ is $\{\phi, X, \{c\}, \{a,c\}, \{c,d\}, \{a,c,d\}$, then $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise totally disconnected, while $(X, \mathscr{T}_1)$ and $(X, \mathscr{T}_2)$ are connected. In contrast to the topological situation, components of bitopological spaces need not be closed. Here, for example, $\{d\}$ is a component of $(X, \mathscr{T}_1, \mathscr{T}_2)$ and $\mathscr{T}_1$ cl$\{d\}=\{c,d\}$, $\mathscr{T}_2$ cl$\{d\}=\{b,d\}$, so that $\{d\}$ is neither $\mathscr{T}_1$ closed nor $\mathscr{T}_2$ closed. Notice that $\{d\}=\mathscr{T}_1$ cl$\{d\}\cap\mathscr{T}_2$ cl$\{d\}$ as required by Theorem F of Pervin [3].

PROPOSITION 2. *If $(X, \mathscr{T}_1, \mathscr{T}_2)$ is a bitopological space such that $(X, \mathscr{T}_1)$ is T_1 and if X has a $\mathscr{T}_1$ open base whose members are also $\mathscr{T}_2$ closed then*

$(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise totally disconnected.

PROOF. Let x and y be distinct points of X. There is a $\mathscr{T}_1$ open set U such that $x \in U$ and $y \notin U$. By hypothesis, there is a basic $\mathscr{T}_1$ open set B which is also $\mathscr{T}_2$ closed such that $x \in B \subset U$. Then $X = B | X \sim B$ is a separation of X which separates x and y.

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