$\overline{T}_n$-SPACES

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A topological space for which any two distinct points can be separated by disjoint closed neighborhoods is said to be a Urysohn space or a $\overline{T}_2$-space. One can easily construct an example of a $\overline{T}_2$-space which is not regular as well as an example of a Hausdorff space which is not $\overline{T}_2$. In this paper we define for every positive integer n a separation property denoted by $\overline{T}_n$ which for $n=1$ corresponds with the Hausdorff separation property and for $n=2$ corresponds with the Urysohn separation property. From the definition below it is obvious that $\overline{T}_{n+1}$ implies $\overline{T}_n$ for each n and that a regular space is $\overline{T}_n$ for every n. We show that for any n there exists a space which is $\overline{T}_n$ but not $\overline{T}_{n+1}$. We also give an example of a space which is $\overline{T}_n$ for every n but which is not regular.

DEFINITION. A space $(X, \mathscr{T})$ is $\overline{T}_n$ if given any two distinct points p, q of X then there exists $O_i \in \mathscr{T}$ $(1 \leq i \leq n)$ with $p \in O_1$, $\overline{O}_i \subset O_{i+1}$ $(1 \leq i < n)$, $q \notin \overline{O}_n$.

THEOREM. *There exists a space which is $\overline{T}_n$ but not $\overline{T}_{n+1}$ for any positive integer n.*

PROOF. Let n be given. Let Z denote the set of positive integers and $\{R_i\}_{i=1}^n$ be a set of disjoint copies of the positive real line. Let x, y be two points not contained in any R_i, $1 \leq i \leq n$.

Case 1. n even. Let $X_1 = R_1 \sim \bigcup_{k=1}^{\infty}(\{4k-1\} \cup \{4k-1-\frac{1}{m} : m \in Z\})$. Let $X_{2r} = R_{2r} \sim \{4m : m \in Z\}$, $1 \leq r \leq \frac{n}{2}$. Let $X_{2r-1} = R_{2r-1} \sim \bigcup_{k=1}^{\infty}\left\{4k - \frac{1}{m} : m \in Z\right\}$, $2 \leq r \leq \frac{n}{2}$. Let $Y = \{x, y\} \cup \bigcup_{i=1}^{n} X_i$.

Topologize Y as follows: Let a neighborhood system for the point x be composed of all sets of the form $\{x\} \cup \{l \in X_1 : l \in \bigcup_{k=k_0}^{\infty} (4k, 4k+1) : k_0 \in Z\}$ and a system for y be all sets of the form $\{y\}\cup\{l\in X_1 : l\in\bigcup_{k=k_0}^{\infty}(4k-2, 4k-1),\ k_0\in Z\}$.

For $1\leq r\leq\frac{n}{2}$, let a neighborhood system for any point of X_{2r-1} which is not of the form $4m, m\in Z$, be composed of all the open subsets of the reals which contain the point but which do not contain any point of the form $4m$. Let a neighborhood system for a point $4m$ in X_{2r-1} be composed of all sets of the form $\left\{l\in X_{2r-1} : 4m-t<l<4m+t\right\}\cup\left\{l\in X_{2r} : 4m-t<l<4m+t\right\}\sim\left\{l\in X_{2r} : l=4m-\frac{1}{s},\ s\in Z\right\}$ for $t\in(0,1)$.

For $1\leq r<\frac{n}{2}$ let a neighborhood system for any point X_{2r} which is not of the form $4m-\frac{1}{s}$; $s,\ m\in Z$, be composed of all the open subsets of the reals which contain the point but which do not contain any point of the form $4m-\frac{1}{s}$: $s, m\in Z$. Let a neighborhood system for a point $4m-\frac{1}{s}$ in X_{2r} be composed of all sets of the form

$$\left\{l\in X_{2r} : 4m-\frac{1}{s}-t<l<4m-\frac{1}{s}+t\right\}\cup\left\{l\in X_{2r+1} : 4m-\frac{1}{s}-t<l<4m-\frac{1}{s}+t\right\}$$

for $t\in(0,1)$.

Let a neighborhood system for any point of X_n which is not of the form $4s-\frac{1}{m}$: $s, m\in Z$, be composed of all the open subsets of the reals which contain the point but which do not contain any point of the form $4s-\frac{1}{m}$: $s,\ m\in Z$. Let a neighborhood system for a point $4s-\frac{1}{m}$ in X_n be composed of all sets of the form

$$\left\{l\in X_n : 4s-\frac{1}{m}-t<l<4s-\frac{1}{m}+t\right\}\cup\left\{l\in X_1 : 4s-1-\frac{1}{m}-t<l<4s-1-\frac{1}{m}+t\right\}$$

for $t\in(0,1)$.

One can show that the neighborhood system described above defines a topology on the set Y. It is easily observed that there does not exist a set of open subsets $\{O_i\}_{i=1}^{n+1}$ with $x\in O_1$, $\overline{O}_i\subset O_{i+1}$ $(1\leq i\leq n)$. $y\notin\overline{O}_{n+1}$ and that for any two distinct points p, q of Y a set $\{O_i\}_{i=1}^{n}$ exists with $p\in O_1$, $\overline{O}_i\subset O_{i+1}$ $(1\leq i<n)$, $q\notin\overline{O}_n$.

That is, Y is $\overline{T}_n$ but not $\overline{T}_{n+1}$.

Case 2. n odd, say $n=2r-1$. Let Y' denote the set corresponding to $2r$ in case 1 and $Y=Y'\sim X_{2r}$. Let the neighborhood system for each point of $Y\sim X_n$ be the same as that defined for the corresponding point in Y' of case 1. Let a neighborhood system for any point of X_n which is not of the form $4m$, $m\in Z$, be composed of all the open subsets of the reals which contain the point but which do not contain any point of the form $4m$. Let a neighborhood system for a point $4m$ in X_n be composed of all sets of the form

$$\{l\in X_n : 4m-t<l<4m+t\}\cup\{l\in X_1 : 4m-1-t<l<4m-1+t\} \text{ for } t\in(0,1).$$

Once more one can show the above construction yields a topological space which is $\overline{T}_n$ but not $\overline{T}_{n+1}$.

REMARK. For an example of a non-regular space which is $\overline{T}_n$ for each n but which is not regular one need only consider the unit interval with topology generated by the standard topology along with the set of rationals.

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