REGULAR SPACES AND FUNCTIONAL SEPARATION

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A topological space M is said to be completely regular iff A is a closed subset and x is a point not in A imply there is a continuous function f from M to the closed unit interval $[0,1]$ such that $f(x)=0$ and $f(y)=1$ for all y in A. The main purpose of this paper is to prove that regular spaces can also be characterized by a similar property.

Weil [3] introduced uniform spaces and generalized the concept of uniform continuity for pseudometric spaces; the topologies of uniform spaces are completely regular. Thampuran [2] has shown that regular spaces can be characterized by a structure which has some similarities to a uniformity and hence the concept of uniform continuity can be generalized to this structure.

Unless otherwise specified the terminology of this paper conforms to that of Kelley [1].

Let M be a set and $\mathscr{T}$ a topology for M. Denote by k the Kuratowski closure function of $\mathscr{T}$ and by (M,k) the topological space. Take $\mathscr{C}A=M-A$ for $A\subset M$. Composition of functions will be denoted by juxtaposition; thus $\mathscr{C}k$ will represent $\mathscr{C}(kA)$ for $A\subset M$. If A consists of a single point x we will write x for A.

DEFINITION 1. Let M be a topological space. Then M is said to be *regular* iff A is a closed subset and x a point not in A imply x and A have disjoint neighborhoods.

DEFINITION 2. A set-valued set-function n from the power set, of M, to itself is said to be a ***neighborhood function*** for M iff for all subsets A, B of M

1. $n\phi=\phi$
2. $A\subset nA$ and
3. $nA\subset nB$ if $A\subset B$.

The ordered pair (M,n) is said to be a ***neighborhood space.*** In a neighborhood space (M,n), a subset A of M is said to be a neighborhood of a point x iff $x\in\mathscr{C}n\mathscr{C}A$.

Let (M,n) be a neighborhood space and A a subset of M. It is easy to show that x is in nA iff every neighborhood of x intersects A.

DEFINITION 3. Let (M,n), (L,p) be two neighborhood spaces and f a function from M to L. We will say f is continuous at a point x of M iff B is a neighborhood of $f(x)$ implies the inverse of B, under f, is a neighborhood of x; f is said to be continuous iff f is continuous at each point of M.

It is easy to show that f is continuous iff $fn\subset pf$.

Let R be the reals. Define a neighborhood function r for the reals as follows. u, v will denote real numbers.

1. $ru=\begin{cases}(1/3,\infty) \text{ if } 1/2<u \\ (1/4,\infty) \text{ if } 1/3<u\leq 1/2 \\ (1/(m+2),\ 1/(m-1)] \text{ if } 1/(m+1)<u\leq 1/m,\ m=3,4,\cdots\cdots \\ (-\infty,0] \text{ if } u\leq 0\end{cases}$
2. $rA=\bigcup\{ru : u\in A\}$ if $A\subset(-\infty,0]\cup(v,\infty)$ for some $0<v$
3. $rA=\bigcup\{ru : u\in A\}\cup r0$ if $\inf\{u : u\in A\}=-\infty$ or ≤ 0.

It is obvious that a set A is a neighborhood of a point u iff

1. $ru\subset A$ for u in $(0,\infty)$ and
2. $(-\infty,1/m)\subset A$ for u in $(-\infty,0]$, for some $m=1,2,3,\cdots\cdots$

LEMMA. *Let (M,k) be a topological space, $S(t)=M$ for $t>1$ and for $t,u=1/m$, $m=1,2,3,\cdots\cdots$, let $S(t)$ be an open set such that $kS(t)\subset S(u)$ when $t<u$. Define a function f from M to the neighborhood space (R,r) by $f(x)=\inf\{t : x\in S(t)\}$ for all x in M. Then f is continuous.*

PROOF. Let $y\in M$. First consider the case where $f(y)$ is in $(1/(m+1),1/m]$, $m=3,4,\cdots\cdots$. It is enough to show that the inverse under f of $(1/(m+2),$ $1/(m-1)]$ is a neighborhood of y. Let $A=\{x : f(x)\leq 1/(m-1)\}$. Then x is in A iff x is in $S(1/(m-1))$ and so $A=S(1/(m-1))$. It is clear that y is in A and so A is a neighborhood of y. Next take $B=\{x :\ f(x)>1/(m+2)\}$. Then x in $\mathscr{C}S(1/(m+2))$ implies $f(x)>1/(m+2)$, since $f(x)\leq 1/(m+2)$ would mean $x\in S(1/(m+2))$, and so $\mathscr{C}S(1/(m+2))\subset B$. Now $y\in\mathscr{C}S(1/(m+1))$ since $y\in S(1/(m+1)$ would mean $f(y)\leq 1/(m+1)$. We also know $S(1/(m+2))\subset kS(1/(m+2))\subset S(1/(m+1))$. Hence it follows $y\in\mathscr{C}S(1/(m+1))\subset\mathscr{C}kS(1/(m+2))\subset\mathscr{C}S(1/(m+2))\subset B$. Therefore B is also a neighborhood of y and so $A\cap B$ is a neighborhood of y. This proves the continuity of f at y.

If $f(y)\leq 0$ then $S(1/m)=\{x :\ f(x)\leq 1/m\}$ is a neighborhood of y for each $m=1,2,\cdots\cdots$. If $f(y)$ is in $(1/3,1/2]$ then $\{x ; f(x)>1/4\}$ is a neighborhood of y and if

$f(y) > 1/2$ then $\{x: f(x) > 1/3\}$ is a neighborhood of y. Hence f is continuous.

THEOREM. *A topological space (M, k) is regular iff A a closed subset and x a point, of M, not in A imply there is a continuous function f from (M, k) to (R, r) such that $f(x)=0$ and f is 1 on A.*

PROOF. Let the space be regular. Take $S(t)=M$ for $t>1$ and $S(1)=\mathscr{C}A$. Since (M, k) is regular we can define by induction open neighborhoods $S(t)$ of x such that $kS(t)\subset S(u)$ if $t<u$ for all t, $u=1/m$, $m=1,2,3,\cdots\cdots$. Take $f(y)=\inf\{t: y\in S(t)\}$, for all y in M.

The converse is obvious.

Instead of taking all the reals R, it is clearly equivalent to take $N=1, 1/2, 1/3, \cdots\cdots, 0$ and define r as follows. Let u denote a member of N and A a subset of N.

$$ru=\begin{cases}\left\{1, \frac{1}{2}\right\} \text{ if } u=1 \\ \{1/(m+1),\ 1/m, 1/(m-1)\} \text{ if } u=1/m,\ m=1/2, 1/3, \cdots\cdots. \\ 0 \text{ if } u=0\end{cases}$$

$rA=\bigcup\{ru: u\in A\}$ if there is a positive integer m such that v in A implies $v=0$ or $1/m<v$ and

$rA=\{0\}\bigcup\{ru: u\in A\}$ if $\inf\{u: u\in A\}=0$

We can also consider the set of all positive integers $1, 2, 3, \cdots\cdots$ together with an entity which is not a positive integer and define r in the obvious way.

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