A NOTE ON MATSUSHIMA FORMULA OF DISCRETE UNIFORM SUBGROUPS OF SEMISIMPLE LIE GROUPS

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1. Introduction.

Let G be a connected semisimple Lie group with finite center and K be a maximal compact subgroup of G. Then $X=G/K$ is a Riemannian symmetric space. Let Γ be a discrete uniform subgroup of G, that is, the quotient space $\Gamma\backslash X$ is compact. Let $\mathscr{G}$ be the Lie algebra of left invariant vector fields on G and $\mathscr{K}$ the subalgebra of $\mathscr{G}$ corresponding to K such that $\mathscr{G}=\mathscr{K}\oplus\mathfrak{M}$ with respect to the killing form on $\mathscr{G}$. In [4], Y. Matsushima has obtained an interesting formula for the Betti numbers of $\Gamma\backslash X$ in terms of multplicities of certain irreducible unitary representations of G in $L^2(\Gamma\backslash G)$. Our purpose is to give an analogous formula for the dimension of the cohomology group $H^p(\Gamma, X, \rho)$, $p\geq 1$, with respect to an arbitrary representation ρ of G in a finite dimensional complex vector space F. When $G=SL(2, R)$, I.M. Gelfand conjectured in [2] that the decomposition of $L^2(\Gamma\backslash G)$ shall give a complete set of invariants for the moduli problem of compact Riemann surfaces. Here, as a consequence of the dimension formula of $H^p(\Gamma, X, \rho)$, we observe that only the irreducible unitary representation of the discrete series of index 4 is essential to $H^1(\Gamma, \mathrm{G})$ (see [5]). In fact, the representation space of the discrete series of index 4 is the space of quadratic differentials in [1]. We shall follow the notation and terminology of [3] and [4].

2. The dimension formula of $H^p(\Gamma, X, \rho)$ $(p\geq l)$.

Let $A^p(\Gamma, X, \rho)$ and $A^p(\Gamma\backslash G, K, \rho)$ be the space of F-valued p-forms on manifolds X and $\Gamma\backslash\mathrm{G}$ defined in [4]. To each element $\eta\in A^p(\Gamma, X, \rho)$, there corresponds an element $\eta^\circ\in A^p(\Gamma\backslash G, K, \rho)$ in a one to one way. Each element $\eta^\circ\in A^p(\Gamma\backslash G, K, \rho)$ can be expressed as

$$\eta^\circ=\frac{1}{p!}\sum_{\lambda_1,\dots\dots,\lambda_p=1}^{n}\eta_{\lambda_1\cdots\lambda_p}\, w^{\lambda_1}\wedge\cdots\wedge w^{\lambda_p},$$

where $\eta_{\lambda_1\cdots\lambda_p}=\eta^\circ(X_{\lambda_1}\cdots X_{\lambda_p})$, $1\leq\lambda_1<\cdots<\lambda_p\leq n$, for a particularly chosen basis

$\{X_1, \cdots, X_n\}$ of $\mathscr{G}$ and its dual basis $\{w^1, \cdots, w^n\}$ (see [4]). Thus, η° can be regarded as an $(F\otimes\Lambda^p\mathfrak{M}^*)$-valued function on $\Gamma\backslash G$, where $\Lambda^p\mathfrak{M}^*$ is the p^{th} exterior product of the dual space $\mathfrak{M}^*$ of $\mathfrak{M}$. The fundamental result in [4] is that every cohomology class in $H^p(\Gamma, X, \rho)$ is represented by a unique harmonic p-form η in $A^p(\Gamma, X, \rho)$, that is,

(2.1) $(\Delta\eta)_{\lambda_1\cdots\lambda_p} = (-C+\rho(C))\eta_{\lambda_1\cdots\lambda_p}$, $p\geq 1$, $1\leq\lambda_1<\cdots<\lambda_p\leq n$,

where C is the Casimir operator and $\rho(C)$ is the Casimir operator of ρ. In particular, if ρ is irreducible, (2.1) becomes $C\eta_{\lambda_1\cdots\lambda_p} = \lambda_\rho\,\eta_{\lambda_1\cdots\lambda_p}$, where λ_p is a constant.

The complex vector space F may be decomposed into a direct sum of irreducible G-submodules $F^{(1)}, \cdots, F^{(t)}$ such that $\rho=\rho^{(1)}\oplus\cdots\oplus\rho^{(t)}$. Further, we let $F^{(i)}=S_1^{(i)}\oplus\cdots\oplus S_{mi}^{(i)}$ (resp. $\Lambda^p\mathfrak{M}^*=V_1^*\oplus\cdots\oplus V_{S_p}^*$) be a decomposition of $F^{(i)}$ (resp. $\Lambda^p\mathfrak{M}^*$) into direct sum of K-submodules so that $\rho^{(i)}\,|\,K=\rho^{(i)}\oplus\cdots\oplus\rho_{m_i}^{(i)}$ and $Ad^{p*}=\tau_1^{p*}\oplus\cdots\oplus\tau_{S_p}^{p*}$, where Ad^{p*} is the representation of K on $\Lambda^p\mathfrak{M}^*$ induced by the adjoint representation Ad of K on $\mathfrak{M}$. We have $F\otimes\Lambda^p\mathfrak{M}^*=\sum_{i,h,j} S_h^{(i)}\otimes V_j^*$. Let $P_{hj}^{(i)}$ be the projection of $F\otimes\Lambda^p\mathfrak{M}^*$ onto the direct factor $S_h^{(i)}\otimes V_j^*$. Then $P_{hj}^{(i)}$ commutes with $(\rho\otimes Ad^{p*})(k)$, for all $k\in K$, and the Laplacian Δ. Consequently, if $\eta\in A^p(\Gamma, X, \rho)$ is harmonic, $P_{hj}^{(i)}\,\eta$ is also harmonic. We easily get $\dim H^p(\Gamma, X, \rho)=\sum_{i=1}^{t}\dim H^p(\Gamma, X, \rho^{(i)})$. Let T be an irreducible unitary representation of G in a Hilbert space H and let $N(T)$ be the multiplicity of T in $L^2(\Gamma\backslash G)$. T_K denotes the restriction of T to K and $M(T_K;\tau)$ denotes the multiplicity in T_K of an irreducible representation τ of K. The domain of the Casimir operator $T(C)$ of T is a dense subspace of H. If T is nontrivial and irreducible, T(C) is a scalar λ_τ-multiple of the identity transformation of the domain of $T(C)$. The set of irreducible unitary representations T of G such that $\lambda_T=\lambda_\rho(i)$ is denoted by $D_\rho\,(i)$. A quite simple modification of the proof in [3] implies

$$\dim\, H^p(\Gamma, X, \rho^{(i)})=\sum_{T\in D_\rho(i)} N(T)\Big[\sum_{h=1}^{m_i}\sum_{j=1}^{S_p} M(T_K;\rho_h^{(i)}\otimes\tau_j^{\rho^*})\Big].$$

Consequently, the dimension formula of $H^p(\Gamma, X, \rho)$ is given by

$$\dim H^p(\Gamma, X, \rho) = \sum_{i=1}^{t} \sum_{T \in D\rho(i)} N(T) \left[\sum_{h=1}^{mi} \sum_{j=1}^{S_p} M(T_K; \rho_h^{(i)} \otimes \tau_j^{p*}) \right].$$

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