POSNER'S THEOREM ON PI RINGS

By Kevin McCrimmon

In [1] S. A. Amitsur showed that a ring without zero divisors which satisfies a polynomial identity has a two-sided ring of quotients which is a division ring of finite dimension over its center. This was extended by E. C. Posner [7] to arbitrary prime PI rings:

THEOREM. *If R is a prime ring satisfying a polynomial identity then R has a two-sided ring of quotients Q which is a simple finite-dimensional algebra over its center.*

Alternate proofs have been given by W. A. Martindale [6], I. N. Herstein [5], and A. W. Goldie [4]. In this note we offer an elementary proof using nothing more than the Density Theorem (see [5, p. 41]).

We assume R is an algebra over some commutative ring Ω satifying a monic polynomial identity of degree d with coefficients in Ω, which we may take to be multilinear of the form

$$p(x_1, \cdots, x_d) = x_d \cdots x_1 + \sum_{\sigma \neq 1} \alpha_\sigma x_{\sigma(d)} \cdots x_{\sigma(1)}.$$

Thus we assume $p(r_1, \cdots, r_d) = 0$ for all $r_1, \cdots, r_d \in R$. A *uniqueness sequence* of length n in R relative to a representation of R on a (right) Ω-module M is a sequence $r_1, \cdots r_n$, of elements in R such that for some $m \in M$ we have $r_n \cdots r_1(m) \neq 0$ but $r_{\sigma(n)} \cdots r_{\sigma(1)}(m) = 0$ for any other permutation $\sigma \neq 1$. Clearly if R satisfies the polynomial identity $p(x_1, \cdots, x_d)$ of degree d there can be no uniqueness sequences of length d in R. There is a standard process, due to Amitsur [2, p. 102–103], for constructing uniqueness sequences.

LEMMA. *If an algebra A has a representation on a (right) vector space V over a field Ω such that*

(*) *for every subspace $W \subset V$ of dimension $< d$ and for every $v \notin W$ there exists an element $a \in A$ with $a(W) = 0$ $a(v) \notin W + v\Omega$*

then there exists a uniqueness sequence in A of length d.

We make no attempt to improve on the demonstration [7, p. 180; 5, p. 184] that a prime PI ring R is a right and left Goldie ring, hence has a two-sided ring of quotients Q of the form $Q=End_D(V)\cong D_n$ for V an n-dimensional right vector space over the division ring D. The difficult part is to show finite-dimensionality of Q (or D) over its center ϕ.

Let Ω be a maximal subfield of D, and let ω_r denote right multiplication on V by the scalar $\omega\in\Omega$. Then $A=R\Omega_r$ is a left (and right) *pre-order* in $B=Q\Omega_r$ in the sense that every $b\in B$ may be written $b=c^{-1}a$ (or ac^{-1})for some a, $c\epsilon A$. (Indeed, if $\sum b=q_i\ \omega_i$ for $q_i\ \epsilon Q$, $\omega_i\ \epsilon\Omega$, then we know $q_i=r^{-1}r_i$ for some r, $r_i\in R\subset A$ since R is a two-sided order in Q. hence $b=c^{-1}a$ for $c=r$, $a=\sum r_i\omega_i\in A$). Here B is a dense ring of linear transformations on the right vector space V over Ω: B acts irreducibly on V since Q already does, and its centralizer consists of those scalar multiplications d_r $(d\in D)$ which commutes with all ω_r, and since Ω is maximal this means $d\in\Omega$. Furthermore, since Ω_r commutes with R, A will satisfy any multilinear polynomial identity that R does. Therefore A satisfies $p(x_1, \cdots\cdots, x_d)$, so A has no unieqeness sequences of length d. By the lemma, (*) must be violated for some finite-dimensional subspace W and some $v\notin W$: whenever $a\in A$ satisfies $a(W)=0$ then necessarily $a(v)\in W+v\Omega$.

We first find a nonzero finite-dimensional subspace V_0 which is invariant under A. If V itself is finite-dimensional we take $V_0=V$, whereas if V is not finite-dimensional then certainly $W+v\Omega$ is not all of V, hence by the density of B on V there exists $b\in B$ with $b(W)=0$ and $b(v)\notin W+v\Omega$ (in particular, $b(v)\neq 0$). Since A is a left pre-order in B we can write $b=c^{-1}\ a_0$ for c, $a_0\in A$. Then any element $a\in Aa_0\subset A$ annihilates W since b does, $a(\mathrm{W})\subset Aa_0\,(W)=Acb(W)=0$; therefore by choice of W and v we cannot have $a(v)\notin W+v\Omega$, so we must have $a(v)\in W+v\Omega$ for all $a\in Aa_0$. But $v_0=a_0(v)=cb(v)$ is nonzero since c is invertible and $b(v)\neq 0$, so $V_0=A(v_0)$ is a nonzero subspace invariant under A which is finite dimensional since $V_0=A(v_0)=Aa_0(v)\subset W+v\Omega$.

We now show the only such invariant subspace is $V_0=V$. This will follow from the following simple observation.

LEMMA. *If A is a right pre-order in a dense ring B of linear transformations on V over a field Ω, then the only finite-dinensional subspaces of V invariant under A are V and* 0.

PROOF. Suppose V_0 is invariant and finite-dimensional. If $V_0\neq 0$, V then by density there is a $b \in B$ with $b(V_0) \not\subset V_0$. Since A is a right pre-order in B we can write $b = ac^{-1}$ for $a, c \in A$. But for invertible c, $c(V_0) \subset V_0$ implies $c(V_0) = V_0 = c^{-1}(V_0)$ by the finite-dimensionality of V_0, and thus $b(V_0) = ac^{-1}(V_0) = a(V_0) \subset V_0$, a contradiction.

Thus $V = V_0$ is finite-dimensional over Ω, and $B = \mathrm{End}_\Omega(V)$ (by density) is also finite-dimensional over Ω. Now $B = Q\Omega_r$ is a nonzero homomorphic image of the central simple algebra $Q \otimes_\Phi \Omega$ over Ω, so the homomorphism must be an isomorphism of Ω-algebras and $\dim_\Phi Q = \dim_\Omega Q \otimes_\Phi \Omega = \dim_\Omega B < \infty$.

REMARK. We note in passing that the Faith-Utumi Theorem [3, p. 57] says that our given order R has $E_n \subset R \subset D_n$ for some two-sided order E in D. E is a *PI* ring (as a copy Ee_{11} of E sits inside R), and it is without zero divizors (clearly), so by Amitsur's original result the ring of quotients D is finite-dimensional over its center. Thus Amitsur implies Posner with the help of Faith and Utumi.

University of Virginia

BIBLIOGRAPHY

[1] S. A. Amitsur, *On rings with identities*, J. London Math. Soc. 30 (1955), 464—470.

[2] S. A. Amitsur, *Rings with involntion*, Israel J. Math. 7(1969) 63—68.

[3] C. Faith and Y. Utumi, *On Noetherian prime rings*, Trans. Amer. Math. Soc. 114 (1965), 53—60.

[4] A. W. Goldie, *A note on prime rings with polynomial identities*, J. London Math. Soc. (2) 1 (1969), 606—608.

[5] I. N. Herstein, *Noncommntative rings*, Carus Mathematical Monographs No 15, Math. Assoc of America.

[6] W. A. Martindale, *Prime rings satifying a generalized polynomial identity*, J. of Algebra 12 (1969), 576—584.

[7] E. C. Posner, *Prime rings satisfying a polynomial identity*, Proc. Amer. Math. Soc. 11 (1960), 180—184.