# ON DOUBLE INTEGRALS INVOLVING MEIJER'S $G$-FUNCTION

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**1.** The object of this paper is to evalute number of integrals involving Meijer's $G$-function in two variables. By proper choice of parameters, the $G$-function reduces to many well known functions such as; Macrobert's $E$-function, Wittaker functions, hypergeometric functions, exponential function (i. e., in Laplace transform) and Bessel functions etc.

In this paper a new method for evaluating integrals is developed. Some authors have obtained integrals for $G$-functions in two variables, with the help of complicated theorems. This method is rather easy and evaluates number of integrals in a neat form, which are difficult to tackle otherwise. The following notation will be used throughout this paper.

$$G^{m,n}_{p,q}\left(st\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right) \text{ stands for } G^{m,n}_{p,q}\left(st\left|\begin{matrix}a_1,\cdots\cdots, & a_p\\ b_1,\cdots\cdots, & b_q\end{matrix}\right.\right)$$

unless ambiguity can arise. For definition and properties of $G$-functions see Erdélyi [1].

**2. Theorem.** *If* $p+q<2(m+n)$, $|\arg z|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi$, $\mathrm{Re}\,(b_j-\alpha)>-1$, $j=1,2,\cdots\cdots,m$, *then*

$$\int_0^\infty\int_0^\infty x^{\alpha-2}y^{-\alpha}(x^2+y^2)e^{-\beta y/x}G^{m,n}_{p,q}\left(\frac{y(x^2+y^2)}{x}\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right)f(x^2+y^2)dydx$$

$$=\beta^{\alpha-1}\int_0^\infty G^{m,n+1}_{p+1,q}\left(\frac{z}{\beta}\left|\begin{matrix}\alpha, a_p\\ b_q\end{matrix}\right.\right)f(z)dz,$$

*provided* $f(z)=0(z^{-\delta})$ *for large* $z$ *and* $f(z)=0(z^{\varepsilon-\frac{1}{2}})$ *for small* $z$: $\delta>0$, $\varepsilon>0$.

PROOF. Since [(2), p. 899], we have

$$\int_0^{\pi/2}(\tan\theta)^{-\alpha}e^{-\beta\tan\theta}G^{m,n}_{p,q}\left(z\tan\theta\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right)\sec^2\theta\, d\,\theta=\beta^{\alpha-1}G^{m,n+1}_{p+1,q}\left(\frac{z}{\beta}\left|\begin{matrix}\alpha, a_p\\ b_q\end{matrix}\right.\right).$$

On putting $z=r^2$, we have

$$\int_0^{\pi/2}\frac{1+\tan^2\theta}{(\tan\theta)^\alpha}e^{-\beta\tan\theta}G^{m,n}_{p,q}\left(r^2\tan\theta\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right)d\,\theta=\beta^{\alpha-1}G^{m,n+1}_{p+1,q}\left(\frac{r^2}{\beta}\left|\begin{matrix}\alpha, a_p\\ b_q\end{matrix}\right.\right).$$

Now multiplying both the sides by $rf(r^2)$ and integrating between the limits

$(0,\infty)$, we have

$$\int_0^\infty rf(r^2)dr\int_0^{\pi/2}\frac{1+\tan^2\theta}{(\tan\theta)^\alpha}e^{-\beta\tan\theta}G^{m,n}_{p,q}\left(r^2\tan\theta \,\middle|\, {a_p \atop b_q}\right)d\theta$$

$$=\beta^{\alpha-1}\int_0^\infty rf(r^2)G^{m,n+1}_{p+1,q}\left(\frac{r^2}{\beta}\,\middle|\,{\alpha, a_p \atop b_q}\right)dr.$$

On putting $x=r\cos\theta$, $y=r\sin\theta$; we get

$$\int_0^\infty\int_0^\infty x^{\alpha-2}y^\alpha(x^2+y^2)e^{-\beta y/x}G^{m,n}_{p,q}\left(\frac{y(x^2+y^2)}{x}\,\middle|\,{a_p \atop b_q}\right)f(x^2+y^2)dydx$$

$$=\beta^{\alpha-1}\int_0^\infty G^{m,n+1}_{p+1,q}\left(\frac{z}{\beta}\,\middle|\,{\alpha, a_p \atop b_q}\right)f(z)dz,$$

which is the required result.

**3. Applications.**

Suppose $f(z)=\dfrac{z^{\rho-1}}{(z+r)^\sigma}$. Therefore we have from the theorem:

$$\int_0^\infty\int_0^\infty\frac{x^{\alpha-2}y^{-\alpha}}{(x^2+y^2+r)^\sigma}(x^2+y^2)^\rho e^{-\beta y/x}G^{m,n}_{p,q}\left(\frac{y(x^2+y^2)}{x}\,\middle|\,{a_p \atop b_q}\right)dydx$$

$$=\beta^{\alpha-1}\int_0^\infty\frac{z^{\rho-1}}{(z+r)^\sigma}G^{m,n+1}_{p+1,q}\left(\frac{z}{\beta}\,\middle|\,{\alpha, a_p \atop b_q}\right)dz.$$

By evaluating the integral on the right side, we get

$$(3.1)\quad\int_0^\infty\int_0^\infty\frac{x^{\alpha-2}y^{-\alpha}}{(r+x^2+y^2)^\sigma}(x^2+y^2)^\rho e^{-\beta y/x}G^{m,n}_{p,q}\left(\frac{y(x^2+y^2)}{x}\,\middle|\,{a_p \atop b_q}\right)dydx$$

$$=\frac{r^{\rho-\sigma}\beta^{\alpha-1}}{\Gamma(\sigma)}G^{m+1,n+2}_{p+2,\,q+1}\left(\frac{r}{\beta}\,\middle|\,{1-\rho,\alpha,a_p \atop \sigma-\rho,\quad b_q}\right),$$

$p+q<2(m+n)$, $\left|\arg\frac{1}{\beta}\right|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi$, $|\arg r|<\pi$,
$R(\rho+b_j>0$, $j=1,\cdots,m$; $R(\rho-\sigma+a_j)<1$, $j=1,\cdots,n$

and under the conditions of the theorem.

By considering

$$f(z)=\begin{cases} z^{-1/2}e^{-\delta\sqrt{z}} \\ z^{-\rho}J_\nu(2\sqrt{z}) \\ z^{-\rho}K_\nu(2\sqrt{z}) \\ z^{-\rho}N_\nu(2\sqrt{z}),\end{cases}$$

we get the following integrals

$$(3.2)\ \int_0^\infty\int_0^\infty x^{\alpha-2}y^{-\alpha}(x^2+y^2)^{1/2}e^{(-\beta y-(\sqrt{x^2+y^2})\delta/x}\ G^{m,n}_{p,q}\Big(\frac{y(x^2+y^2)}{x}\ \Big|\ {a_p \atop b_q}\Big)dydx$$

$$=\frac{\beta^{\alpha-1}}{\delta\sqrt{\pi}}G^{m,n+3}_{p+3,q}\Big(\frac{4}{\beta\delta^2}\ \Big|\ {0,\frac{1}{2},\alpha,a_p \atop b_q}\Big),\qquad |\arg\delta|<\frac{\pi}{2},b_j>-\frac{1}{2}$$

along with the conditions of the theorem.

$$(3.3)\ \int_0^\infty\int_0^\infty x^{\alpha-2}y^{-\alpha}(x^2+y^2)^{1-\rho}\exp\Big[-\frac{\beta y}{x}\Big]J_v(2\sqrt{x^2+y^2})$$

$$\times G^{m,n}_{p,q}\Big(\frac{y(x^2+y^2)}{x}\ \Big|\ {a_p \atop b_q}\Big)dydx$$

$$=\beta^{\alpha-1}G^{m,n+2}_{p+3,q}\Big(\frac{1}{\beta}\ \Big|\ {\rho-v/2,\alpha,a_p,\rho+v/2 \atop b_q}\Big),$$

$$-\frac{3}{4}+\max_{1\leqq j\leqq n}R(a_j)<R(\rho)<\min_{1\leqq j\leqq m}R(b_j)+\frac{1}{2}|R(v)|+1$$

and the conditions of the theorem.

$$(3.4)\ \int_0^\infty\int_0^\infty x^{\alpha-2}y^{-\alpha}(x^2+y^2)^{1-\rho}\exp\Big(-\frac{\beta y}{x}\Big)K_v(2\sqrt{x^2+y^2})$$

$$\times G^{m,n}_{p,q}\Big(\frac{y(x^2+y^2)}{x}\ \Big|\ {a_p \atop b_q}\Big)dydx$$

$$=\frac{1}{2}\beta^{\alpha-1}G^{m,n+3}_{p+3,q}\Big(\frac{1}{\beta}\ \Big|\ {\rho-v/2,\rho+v/2,\alpha,a_p \atop b_q}\Big),$$

$$R(\rho)<1-\frac{1}{2}|R(v)|+\min_{1\leqq j\leqq m}R(b_j)$$

and the conditions of the theorem.

$$(3.5)\ \int_0^\infty\int_0^\infty x^{\alpha-2}y^{-\alpha}(x^2+y^2)^{1-\rho}\exp\Big(-\frac{\beta y}{x}\Big)N_v(2\sqrt{x^2+y^2})$$

$$\times G^{m,n}_{p,q}\Big(\frac{y(x^2+y^2)}{x}\ \Big|\ {a_p \atop b_q}\Big)dydx$$

$$=\beta^{\alpha-1}G^{m,n+3}_{p+4,q+1}\Big(\frac{1}{\beta}\ \Big|\ {\rho-v/2,\rho+v/2,\alpha,a_p,\rho+v/2+1/2 \atop b_q,\rho+v/2+1/2}\Big),$$

$$-\frac{3}{4}+\max_{1\leqq j\leqq n}R(a_j)<R(\rho)<\min_{1\leqq j\leqq m}R(b_j)+\frac{1}{2}|R(v)|+1.$$

Again if we consider

$$f(z)=\begin{cases} z^{-1/4}H_v(y'\sqrt{z}) \\ z^r e^{-z}L_k^\sigma(z) \\ z^{(1-r)/n'-1}W_{k,m'}(z^{-1/2n})W_{-k,m'}(z^{-1/2n}) \\ z^{(1-\lambda)/n'-1}K_{2\mu}(z^{-1/2n})K_{2v}(z^{-1/2n}) \\ z^{r+\sigma-1}{}_2F_1(a,b;c;-z), \end{cases}$$

we get the following integrals.

$$(3.6)\int_0^\infty\int_0^\infty x^{\alpha-2}y^{-\alpha}(x^2+y^2)^{3/4}\exp\left(-\frac{\beta y}{x}\right)H_v(\delta\sqrt{x^2+y^2})$$

$$\times G_{p,q}^{m,n}\left(\frac{y(x^2+y^2)}{x}\,\middle|\,\begin{matrix}a_p\\b_q\end{matrix}\right)dydx=2(2\delta/\beta)^{-1/2}\beta^{\alpha-1}G_{q+1,p+4}^{n+2,m+1}\left(\frac{\delta^2\beta}{4}\,\middle|\,\begin{matrix}3/4+v/2,\ 1/2-b_q\\3/4+v/2,\ 1/2-\alpha,\ 1/2-a_p,\ 1/4+v/2,\ 1/4-v/2,\end{matrix}\right),$$

$$R(a_j)<\min(1,3/4-v/2),\quad R(2b_j+v)>-5/2.$$

$$(3.7)\int_0^\infty\int_0^\infty x^{\alpha-2}y^{-\alpha}(x^2+y^2)^{1+r}\exp\left(-\frac{\beta y}{x}-x^2-y^2\right)L_k^\sigma(x^2+y^2)$$

$$\times G_{p,q}^{m,n}\left(\frac{y(x^2+y^2)}{x}\,\middle|\,\begin{matrix}a_p\\b_q\end{matrix}\right)dydx=\frac{(-1)^k}{k!}\beta^{\alpha-1}G_{p+3,q+1}^{m,n+3}\left(\frac{1}{\beta}\,\middle|\,\begin{matrix}-r,\sigma-r,\alpha,a_p\\b_q,\sigma-r+\kappa\end{matrix}\right),$$

$R(r+1+b_j)>-1,\ j=1,\ \cdots,\ m;\ p+q<2(m+n),\ \left|\arg\left(\frac{1}{\beta}\right)\right|<\left(m+n-\frac{p}{2}-\frac{q}{2}\right)\pi.$

$$(3.8)\int_0^\infty\int_0^\infty \frac{x^{\alpha-2}y^{-\alpha}e^{-\beta y/x}}{(x^2+y^2)^{(r-1)/n'}}W_{k,m'}[(x^2+y^2)^{-1/2n'}]W_{-k,m'}[(x^2+y^2)^{-1/2n'}]$$

$$\times G_{p,q}^{m,n}\left(\frac{y(x^2+y^2)}{x}\,\middle|\,\begin{matrix}a_p\\a_q\end{matrix}\right)dydx=(2\pi)^{\frac{1}{2}-n'}(2n')^{2r+\frac{1}{2}}\beta^{\alpha-1}$$

$$\times G_{p+2n'+1,q+4n'}^{m+4n',n+1}\left(\frac{1}{\beta(2n')^{2n'}}\,\middle|\,\begin{matrix}\alpha,(a_p),\Delta[n';r+k+1],\Delta[n';r-k+1]\\\Delta[2n';2r+1],\Delta[n';r+m'+1/2],\Delta[n';r-m'+1/2],(b_q)\end{matrix}\right),$$

where $\Delta[n',\alpha]$ means $n'$ parameters $\frac{\alpha}{n'},\ \frac{\alpha+1}{n'},\ \cdots\cdots,\ \frac{\alpha+n'-1}{n'}$.

The conditions of validity are

(i) $p+q<2(m+n)$ and $\left|\arg\frac{1}{\beta}\right|<\pi\left(m+n-\frac{p}{2}-\frac{q}{2}\right)$,

(ii) $R\left[r\pm m'-\frac{1}{2}(1-a_j)\right]>0,\ R[r+n'(1-a_j)]>0,\ j=1,2,\ \cdots,\ n.$

$$(3.9)\int_0^\infty\int_0^\infty \frac{x^{\alpha-2}y^{-\alpha}}{(x^2+y^2)^{(\lambda-1)/n'}}e^{-\beta y/x}K_{2\mu}[(x^2+y^2)^{-1/2n}]K_{2v}[(x^2+y^2)^{-1/2n}]$$

$$\times G^{m,n}_{p,q}\left(\frac{y(x^2+y^2)}{x}\middle|\begin{matrix}a_p\\ b_q\end{matrix}\right)dydx=\pi^{\frac{3}{2}-n'}2^{-n'}(n')^{2\lambda-3/2}\beta^{\alpha-1}$$

$$\times G^{m+4n',n+1}_{p+2n'+1,q+4n'}\left(\frac{(n')^{-2n'}}{\beta}\middle|\begin{matrix}\alpha,(a_p),\Delta[2n';2\lambda]\\ \Delta[n';\lambda+\mu+v],\Delta[n';\lambda-\mu+v],\Delta[2n';\lambda+\mu-v],b_q\end{matrix}\right).$$

The condition of validity are:

(i) $p+q<2(m+n)$, $\left|\arg\frac{1}{\beta}\right|<\left(m'+n'-\frac{p}{2}-\frac{q}{2}\right)\pi$,

(ii) $R[\lambda\pm\mu\pm v+n'(1-a_j)]>0$, $(j=1,2,\cdots,n')$.

$$(3.10)\int_0^\infty\int_0^\infty x^{\alpha-2}y^{-\alpha}(x^2+y^2)^{\rho+\sigma}\exp\left(-\frac{\beta y}{x}\right){}_2F_1[a,b;c;-x^2-y^2]$$

$$\times G^{m,n}_{p,q}\left(\frac{y(x^2+y^2)}{x}\middle|\begin{matrix}a_p\\ b_q\end{matrix}\right)dydx=\beta^{\alpha-1}\sum_{s=0}^{\infty}\frac{1}{s!}G^{m+2,n+1}_{p+3,q+2}\left(\frac{1}{\beta}\middle|\begin{matrix}\alpha,a_p,\sigma,a+b-c+\sigma+s\\ a-c+\sigma+s,b-c+\sigma+s,b_q\end{matrix}\right),$$

where $p+q<2(m+n)$, $\left|\arg\frac{1}{\beta}\right|<\left(m+n-\frac{p}{2}-\frac{q}{2}\right)\pi$, $R(c)>0$, $R(a-c+\sigma-a_j)>-1$,

$R(b-c+\sigma-a_j)>-1$, $j=1,2,\cdots,n$.

By considering $f(z)$ to be equal to;

(i) $z^{-u/2-1}J_v(\sqrt{z})J_w(\sqrt{z})$,

(ii)
$$z^{\lambda-1}S\left[\begin{matrix}\begin{bmatrix}m_1, & 0\\ p_1-m_1, & q_1\end{bmatrix}\\ \begin{pmatrix}m_2, & n_2\\ p_2-m_2, & q_2-n_2\end{pmatrix}\\ \begin{pmatrix}m_3, & n_3\\ p_3-m_3, & q_3-n_3\end{pmatrix}\end{matrix}\middle|\begin{matrix}a_1,\cdots,a_{p_1};b_1,\cdots,b_{q_1}\\ c_1,\cdots,c_{p_2};d_1,\cdots,d_{q_2}\\ e_1,\cdots,e_{p_3};f_1,\cdots,f_{q_3}\end{matrix}\middle|\begin{matrix}b\\ \\ cz\end{matrix}\right]$$

(iii)
$$z^{s-1}F\left[\begin{matrix}\lambda\\ \mu\\ \nu\\ \sigma\end{matrix}\middle|\begin{matrix}\alpha_1,\cdots,\alpha_\lambda\\ \beta_1;\beta_1';\cdots,\beta_\mu;\beta_\mu'\\ \gamma_1,\gamma_2,\cdots,\gamma_\nu\\ \delta_1;\delta_1';\cdots,\delta_\sigma;\delta_\sigma'\end{matrix}\middle|\begin{matrix}cz\\ \\ \\ dz\end{matrix}\right],$$

(iv) $z^{mk-1}G^{\lambda,\mu}_{\nu,\varepsilon}\left(z^n\middle|\begin{matrix}\alpha_\nu\\ \beta_\varepsilon\end{matrix}\right)$,

we get the following integrals.

$$(3.11)\int_0^\infty\int_0^\infty \frac{x^{\alpha-2}y^{-\alpha}}{(x^2+y^2)^{u/2}}\exp\left(-\frac{\beta y}{x}\right)J_v(\sqrt{x^2+y^2})\,J_w(\sqrt{x^2+y^2})$$

$$\times G_{p,q}^{m,n}\left(\frac{y(x^2+y^2)}{x}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)dydx=\frac{\beta^{\alpha-1}}{\sqrt{\pi}}G_{p+5,q+2}^{m+2,n+2}\left(\frac{4}{\beta}\,\middle|\,\begin{matrix}1+(u-w-v)/2,\alpha,a_p,\ 1+(u+w+v)/2,1+(u-w-v)/2\\ u/2+1/2,u/2+3/2,\ b_q\end{matrix}\right),$$

where $p+q<2(m+n)$, $\left|\arg\frac{1}{\beta}\right|<\left(m+n-\frac{p}{2}-\frac{q}{2}\right)\pi$, $R(w-u+2b_j)>0$, $j=1,2,\cdots,m$; $R(2a_j-u)<3$, $j=1,2,\cdots,n$.

$$(3.12)\quad\int_0^\infty\int_0^\infty x^{\alpha-2}y^{-\alpha}(x^2+y^2)^\lambda\exp\left(-\frac{\beta y}{x}\right)G_{p,q}^{m,n}\left(\frac{y(x^2+y^2)}{x}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)$$

$$\times S\left[\begin{matrix}\begin{bmatrix}m_1, & 0\\ p_1-m_1, & q_1\end{bmatrix}\\ \begin{pmatrix}m_2 & n_2\\ p_2-m_2, & q_2-n_2\end{pmatrix}\\ \begin{pmatrix}m_3, & n_3\\ p_3-m_3, & q_3-n_3\end{pmatrix}\end{matrix}\,\middle|\,\begin{matrix}a_1,\cdots,a_{p_1};b_1,\cdots,b_{q_1}\\ c_1,\cdots,c_{p_2};d_1,\cdots,d_{q_2}\\ e_1,\cdots,e_{p_3};f_1,\cdots,f_{q_3}\end{matrix}\,\middle|\,\begin{matrix}b\\ \\ c(x^2+y^2)\end{matrix}\right]dydx$$

$$=\beta^{\alpha+\lambda-1}S\left[\begin{matrix}\begin{bmatrix}m_1, & 0\\ p_1-m_1, & q_1\end{bmatrix}\\ \begin{pmatrix}m_2, & n_2\\ p_2-m_2, & q_2-n_2\end{pmatrix}\\ \begin{pmatrix}m+m_3, & 1+n+n_3\\ p_3-m_3+q-m, & \\ q_3-m_3+p-n & \end{pmatrix}\end{matrix}\,\middle|\,\begin{matrix}a_1,\cdots,a_{p_1};b_1,\cdots,b_q\\ c_1,\cdots,c_{p_2};d_1,\cdots,d_{q_2}\\ e_1,\cdots,e_{p_3};\Gamma(1-b_q-\lambda)^*;\\ f_1,\cdots,f_{q_3},\ \Gamma(1-\alpha-\lambda),\ \Gamma(1-a_p-\lambda)^*\end{matrix}\,\middle|\,\begin{matrix}b\\ \\ c\beta\end{matrix}\right],$$

where $\Gamma(1-b_q-\lambda)^*=\Gamma(1-b_1-\lambda),\ \Gamma(1-b_2-\lambda),\ \cdots,\ \Gamma(1-b_q-\lambda)$,

$\Gamma(1-a_p-\lambda)^*=\Gamma(1-a_1-\lambda),\ \Gamma(1-a_2-\lambda),\ \cdots,\ \Gamma(1-a_p-\lambda)$,

$0<n_3<q_1+q_3$, $m_1+m_3+n_3>\frac{1}{2}(p_1+q_1+p_3+q_3)$,

$|\arg c|<\left(m_1+m_3+n_3-\frac{p_1}{2}-\frac{q_1}{2}-\frac{p_3}{2}-\frac{q_3}{2}\right)\pi$, $0<m<q$, $0<n<p$,

$m+n>\frac{1}{2}(p+q)$, $\left|\arg\frac{1}{\beta}\right|<\left(m+n-\frac{p}{2}-\frac{q}{2}\right)\pi$,

$R(\lambda+b_j+f_h)>0$, $j=1,2,\cdots,m$, $h=1,2,\cdots,n_3$,

$R(\lambda+a_j+e_h-\alpha_k)<2$; $j=1,2,\cdots,n$, $h=1,2,\cdots,m_3$, $k=1,2,\cdots,m_1$,

$0<m_1+m_3<p_1+p_3.$

(3.13) $$\int_0^\infty\int_0^\infty x^{\alpha-2}y^{-\alpha}(x^2+y^2)^s \exp\left(-\frac{\beta y}{x}\right)G_{p,q}^{m,n}\left(\frac{y(x^2+y^2)}{x}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)$$

$$\times F\left[\begin{matrix}\lambda\\ \mu\\ \nu\\ \sigma\end{matrix}\,\middle|\,\begin{matrix}\alpha_1,\ \cdots,\ \alpha_\lambda\\ \beta_1;\beta'_1,\ \cdots,\ \beta_\mu;\beta'_\mu\\ \gamma_1,\ \cdots,\ \gamma_\nu\\ \delta_1;\delta'_1;\cdots;\delta_\sigma,\ \delta'_\sigma\end{matrix}\,\middle|\,\begin{matrix}c(x^2+y^2)\\ \\ d(x^2+y^2)\end{matrix}\right]dydx$$

$$=\beta^{\alpha+s-1}F\left[\begin{matrix}\lambda+q\\ \mu\\ \nu+p+1\\ \sigma+p+1\end{matrix}\,\middle|\,\begin{matrix}\alpha_\lambda,\ b_q+s\\ \beta_\mu;\beta'_\mu\\ \gamma_\nu,\ \alpha+s,\ a_p+s\\ \delta_\sigma:\delta'_\sigma\end{matrix}\,\middle|\,\begin{matrix}c\beta\\ \\ d\beta\end{matrix}\right].$$

where $b_q+s$ means $q$ parameters $b_1+s,\ b_2+s,\ \cdots,\ b_q+s$ and similarly for other parameters.

The conditions of validity are;

$p+q<2(m+n)$, $\left|\arg\frac{1}{\beta}\right|<\left(m+n-\frac{p}{2}-\frac{q}{2}\right)\pi$, $\lambda+\mu\leqq\nu+\sigma+1$, $-\min_{1\leqq j\leqq m}R(b_j)$
$<R(s)<1-\max_{1\leqq j\leqq m}R(a_j)$,

(3.14) $$\int_0^\infty\int_0^\infty x^{\alpha-2}y^{-\alpha}(x^2+y^2)^{m'k}\exp\left(-\frac{\beta y}{x}\right)G_{p,q}^{m,n}\left(\frac{y(x^2+y^2)}{x}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)$$

$$\times G_{\nu,\varepsilon}^{\lambda,\mu}\left((x2+y^2)^{n'}\,\middle|\,\begin{matrix}\alpha_\nu\\ \beta_\varepsilon\end{matrix}\right)dydx=-\beta^{\alpha-1-m'k}(2\pi)^{\frac{1}{2}(1-n')(1+2n+2m-p-q)/2}$$

$$\times(n')^{m'k(1+p-q)-(1-2m-2n+p+q)/2-\Sigma(a_s)+\Sigma(b_s)}$$

$$\times G_{\nu+n'q,\,\varepsilon+n'+n'p}^{\lambda+n'+nn',\,\mu+n'm}\left(\frac{(n')^{n'(q-p-1)}}{\beta^{n'}}\,\middle|\,\begin{matrix}\Delta_{n'}[-n';1+b_m+m'k-n'],\alpha_\nu,\\ \Delta[n';1-a_n+m'k],\ \Delta[n';m'k],\\ \Delta_{n'}[-n';2+m'k-n'],\ \Delta[n';m'k-(b_{q-m})+1]\\ \beta_\varepsilon,\ \Delta_{n'}[-n';(a_{p-n})-m'k-n']\end{matrix}\right),$$

provided

(i) $0\leqq\lambda\leqq\varepsilon$, $1\leqq\mu\leqq\nu$, $n\geqq0$, $p-n\geqq0$, $q-m\geqq0$, $\nu<\varepsilon$, $1+p<q$,
$R[m'k+n'\beta_j+b_m]>0$ $(j=1,2,\cdots,\ \lambda;\ k=1,2,\cdots,m)$

and $R[m'k+n'a_j-n'+\max\{1,(a_k)\}-1]<0$ $(j=1,2,\cdots,\mu;\ k=1,2,\cdots,n)$

(ii)* $1 \leqq \lambda \leqq \varepsilon,\ 0 \leqq \mu \leqq \nu,\ n \geqq 0,\ p-n \geqq 0,\ q-m \geqq 0,\ \nu > \varepsilon,\ 1+p < q,$

$$R\,[m'k-1-n'+n'a_j+\max\{1, a_k\}] < 0 (j=1,2,\cdots,\mu; k=1,2,\cdots,n)$$

and $R\,[m'k+b_j+n'\beta_k] > 0\ (j=1,2,\cdots,m;\ k=1,2,\cdots,\lambda),\ \alpha=1.$

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* A variety of similar sets of conditions of validity can be worked out by taking different suitable restrictions on the parameters.