NOTES ON SUBMANIFOLDS OF CODIMENSION 2 IN ALMOST CONTACT MANIFOLD

By Un Kyu Kim

1. Introduction

Recently, D.E. Blair, G.D. Ludden and K. Yano [1] obtained the conditions in order that the imbedded submanifold of codimension 2 in an almost complex manifold be almost complex. On the other hand, K. Yano and S. Ishihara [2] have shown that any invariant submanifold of codimension 2 in a contact Riemannian manifold is also a contact Riemannian manifold. By analogous method of [1] we obtain the conditions in order that the imbedded submanifold of codimension 2 of almost contact manifold be almost contact manifold. And also we obtain some properties for the hypersurface of almost contact manifold.

2. Submanifolds of codimension 2 of almost contact manifold

Let (ϕ, ξ, η', G) be an almost contact metric structure of a $(2n+1)$-dimensional almost contact manifold M^{2n+1}, that is,

$$\phi\xi=0, \qquad \phi^2=-I+\eta'\otimes\xi, \qquad \eta'\phi=0,$$
$$G(\xi,\overline{X})=\eta'(\overline{X}), \quad \eta'\xi=1, \quad G(\phi\overline{X},\phi\overline{Y})=G(\overline{X},\overline{Y})-\eta'(\overline{X})\eta'(\overline{Y}),$$

where $\overline{X}$ and $\overline{Y}$ are vector fields on M^{2n+1}. Supppose that N^{2n-1} is an imbedded submanifold of class C^∞ with unit normals C and D and induced metric g. Thus, if B denote the differential of the imbedding and X and Y tangent vector fields on N^{2n-1}, then

$$G(BX,BY)=g(X,Y), \; G(C,C)=1, \quad G(D,D)=1,$$
$$G(C,D)=0, \quad G(BX,C)=0, \; G(BX,D)=0.$$

It is easy to see that we can define a tensor field f of type $(1,1)$, vector fields E, A and F, 1-forms η, α and δ, and functions λ, β and γ on N^{2n-1} by

$$\begin{aligned} &\phi BX=BfX+\eta(X)C+\alpha(X)D, && \xi=BF+\beta C+\gamma D \\ (2,1) \qquad &\phi C=-BE+\lambda D && \delta(X)=\eta'(BX) \\ &\phi D=-BA-\lambda C. && \end{aligned}$$

LEMMA 1. $f, E, A, F, \eta, \alpha, \delta, \lambda, \beta, \gamma$ satisfy

(1) $f^2=-I+\alpha\otimes A+\eta\otimes E+\delta\otimes F$, (2) $\eta f=\beta\delta+\lambda\alpha$, (3) $\alpha f=\gamma\delta-\gamma\eta$,
(4) $\delta f=-\beta\eta-\gamma\alpha$, (5) $f(F)=\beta E+\gamma A$, (6) $\eta(F)=\gamma\lambda$, (7) $\alpha F=-\beta\lambda$,
(8) $\delta(F)=1-\beta^2-\gamma^2$, (9) $f(E)=-\beta F-\lambda A$, (10) $\eta(E)=1-\beta^2-\lambda^2$,
(11) $\alpha(E)=-\beta\gamma$, (12) $\delta(E)=\lambda\gamma$, (13) $f(A)=-\gamma F+\lambda E$, (14) $\eta(A)=-\gamma\beta$,
(15) $\alpha A=1-\gamma^2-\lambda^2$, (16) $\delta(A)=-\lambda\beta$.

PROOF. Computing $\phi^2 BX$ we have

$$Bf^2X+\eta(fX)+\alpha(fX)D-\eta(X)BE+\lambda\eta(X)D-\alpha(X)BA-\lambda\alpha(X)C$$
$$=-BX+\delta(X)BF+\beta\delta(X)C+\gamma\delta(X)D.$$

Comparing tangential and normal parts we obtain (1), (2) and (3). Since $\delta(fX)=\eta'(BfX)=\eta'(\phi BX-\eta(X)C-\alpha(X)D)=-\eta(X)\eta'(C)-\alpha(X)\eta'(D)$, $\eta'(C)=G(\xi,C)=\beta$ and $\eta'(D)=G(\xi,D)=\gamma$, we have (4). Similarly, computing ϕBF we get

$$BfF+\eta(F)C+\alpha(F)D=\beta BE-\beta\lambda D+\gamma BA+\gamma\lambda C.$$

Comparing tangential and normal parts we obtain (5), (6) and (7). Since $\delta(F)=\eta'(BF)=\eta'(\xi-\beta C-\gamma D)=1-\beta^2-\gamma^2$, $\delta(E)=\eta'(BE)=\eta'(-\phi C+\lambda D)=\lambda\gamma$ and $\delta(A)=\eta'(BA)=\eta'(-\phi C+\lambda C)=-\lambda\beta$, we obtain (8), (12) and (16). From $\phi^2 C$ and $\phi^2 D$ we obtain

$$-C+\eta'(C)BF+\eta'(C)\beta C+\eta'(C)\gamma D=-BfE-\eta(E)C-\alpha(E)D-\lambda BA-\lambda^2C \text{ and}$$
$$-D+\eta'(D)BF+\eta'(D)\beta C+\eta'(D)\gamma D=-BfA-\eta(A)C-\alpha(A)D+\lambda BE-\lambda^2D.$$

Similarly, comparing tangential and normal parts we have the remaining identities.

THEOREM 1. *Let $M^{2n+1}(\phi,\xi,\eta',G)$ is an almost contact metric manifold and let N^{2n-1} is an imbedded submanifold of M^{2n+1}. Then we have the following*:

(1) $\beta=0$ *and* $\gamma=\pm 1$ *if and only if N^{2n-1} has an almost contact structure* (f,E,η).

(2) $\beta=\pm 1$ *and* $\gamma=0$ *if and only if N^{2n-1} has an almost contact structure* (f,A,α)

(3) λ *is identically* $+1$ *or* -1 *if and only if N^{2n-1} has an almost contact stucture* (f,F,δ).

PROOF. (1) Suppose that $\beta=0$ and $\gamma=\pm 1$. Then $G(\phi D,\phi D)=G(D,D)-\eta'(D)\eta'(D)=1-\gamma^2=0$ and hence we have $A=0$ and $\lambda=0$. From the lemma 1 we have $\eta(E)=1$ and $\delta(F)=0$, and hence we get $F=0$ from $G(BF,BF)=G(\phi BF,\phi BF)+\delta(F)\delta(F)=0$. Thus we have $f^2=-I+\eta\otimes E$, $\eta(E)=1$, $fE=-BF-\lambda A=0$ and $\eta f=0$, that is, (f,E,η) is an almost contact structure on N^{2n-1}.

Conversely, if (f, E, η) is an almost contact structure on N^{2n-1}, that is, $fE=0$, $\eta f=0$, $\eta E=1$ and $f^2=-I+\eta\otimes E$, then β and λ are all zero from (10) of lemma 1. Since $\alpha\otimes A+\delta\otimes F$ is zero from lemma 1, we have $\alpha(A)A+\delta(A)F=\alpha(A)A=(1-\gamma^2)A=0$. Therefore we find $1-\gamma^2=0$ or $A=0$. If $A=0$, then $\alpha(A)=1-\gamma^2=0$ and hence $\gamma=\pm1$.

(2) and (3) are proved similarly as (1).

REMARKS. (a) There does not exist a submanifold of codimension 2 such that $\xi=BF\pm C$ or $\xi=BF\pm D$ for a non zero vector field F on it.

(b) $\lambda=\pm1$ is the necessary and sufficient condition in order that N^{2n-1} be an invariant submanifold. We obtain this by the similar method of the proof in [2].

A tensor field f of type (1,1) being of constant rank r such that $f^3+f=0$ is called an f-structure of rank r [4].

THEOREM 2. *The tensor f in (2,1) defines an f-structure if and only if λ is identically ±1 or ξ is a normal vector field along N^{2n-1}.*

PROOF. If λ is identically 1 or -1, then N^{2n-1} has an almost contact structure. Hence the tensor f defines an f-structure. If ξ is a normal vector along N^{2n-1}, then we have $F=0$ in (2,1). Hence we get $0=\delta(F)=1-\beta^2-\gamma^2$, $0=\eta(F)=\gamma\lambda$ and $0=\alpha(F)=-\beta\lambda$ from lemma 1. Therefore we have the three cases for β and γ;

(I) $\beta=\pm1$, $\gamma=0$, (II) $\beta=0$, $\gamma=\pm1$ (III) $\beta\neq0$, $\gamma\neq0$.

In all cases, we can say that $F=0$ implies $\lambda=0$. In cases (I) and (II) N^{2n-1} always carries an almost contact structure from theorem 1. In case (III), we have $E\neq0$ and $A\neq0$ from $G(BE, BA)=G(\phi C, \phi D)=-\beta\gamma\neq0$. *If* E is proportional to A then f is of rank $2n-2$ and if E is not proportional to A then f is of rank $2n-3$. On the other hand if ξ is a normal vector field along N^{2n-1} then we have $f^3+f=0$ from lemma 1. Conversely, it suffices to prove that λ is not identically ±1 and ξ is not a normal vector field, then the tensor f is not an f-structure. Let us consider the case $\phi C=-BE$, $\phi D=-BA$ and $\xi=BF+\beta C$. where F is not a zero vector field and β is not zero on N^{2n-1}. By remark (a) β is not ±1. So f is not an f-structure. This completes the proof of this theorem.

Now let us apply the Gauss-Weingarten equations

$$(\nabla_X B)=h(X,Y)C+k(X,Y)D,$$
$$(\tilde{\nabla}_{BX}C)=-BHX+l(X)D,$$
$$(\tilde{\nabla}_{BX}D)=-BKX-l(X)C,$$

where h and k are the second fundamental forms, H and K are the corresponding Weingarten maps, l is the third fundamental form. Moreover, we now assume that the ambient space M is cosymplectic, that is, ϕ and η' are covariant constant with respect to the Riemannian connection of G. Thus we have

$$\tilde{\nabla}_{BX}\phi BY=-h(X,Y)BE+h(X,Y)\lambda D-k(X,Y)BA-k(X,Y)\lambda C+\phi B\nabla_X Y$$

On the other hand,

$$\tilde{\nabla}_{BX}\phi BY=\tilde{\nabla}_{BX}(BfY+\eta(Y)C+\alpha(Y)D)=h(X,fY)C+k(X,fy)D+B(\nabla_X f)Y$$
$$+Bf\nabla_X Y+(\nabla_X\eta)(Y)C+\eta(\nabla_X Y)C+\eta(Y)(-BHX+l(X)D)$$
$$+(\nabla_X\alpha)(Y)D+\alpha(\nabla_X Y)D+\alpha(Y)(-BKX-l(X)C).$$

Therefore, using (2, 1) and comparing tangential part we have

$$(2,2)\qquad -h(X,Y)E-k(X,Y)A=(\nabla_X f)Y-\eta(Y)HX-\alpha(Y)KX.$$

For the induced metric g on N^{2n-1} we have the following

LEMMA 2.
$$g(X,Y)=g(fX,fY)+\eta(X)\eta(Y)+\alpha(X)\alpha(Y)+\delta(X)\delta(Y)$$
$$g(X,fY)=-g(fX,Y)+\delta(Y)g(fX,F)+\beta\eta(X)\delta(Y)+\gamma\alpha(X)\delta(Y)$$
$$\eta(X)=g(X,E)-\delta(X)g(E,F)+\gamma\lambda\delta(X)$$
$$\alpha(X)=g(X,A)-\delta(X)g(F,A)-\beta\lambda\delta(X).$$

THEOREM 3. *Let N^{2n-1} be a submanifold of a cosymplectic manifold M^{2n+1}. If $\lambda\neq\pm1$ and $\beta=\gamma=0$, then f is covariant constant if and only if h and k have the following forms*

$$h=\sigma_1\eta\otimes\eta+\sigma_2(\alpha\otimes\eta+\eta\otimes\alpha)+\sigma_3\alpha\otimes\alpha,$$
$$k=\sigma_2\eta\otimes\eta+\sigma_3(\alpha\otimes\eta+\eta\otimes\alpha)+\sigma_4\alpha\otimes\alpha,$$

where $(1-\lambda^2)^2\sigma_1=h(E,E)$, $(1-\lambda^2)^2\sigma_2=h(E,A)=k(E,E)$

$$(1-\lambda^2)^2\sigma_3=h(A,A)=k(A,E),\quad (1-\lambda^2)\sigma_4=k(A,A)$$

PROOF. It is easily proved by the same method as the theorem 1.7 of [1].

3. Hypersurfaces of almost contact manifold

Let $M(\phi,\xi,\eta',G)$ be a $(2n+1)$-dimensional almost contact metric manifold

M^{2n-1}. Now let N^{2n} be an imbedded hypersurface with unit normal C and let B denote the differential of the imbedding. Define a tensor field f of type $(1,1)$, vector fields E and A, 1-forms η, α and a function λ by

$$\phi BX = BfX + \eta(X)C,\ \xi = BF + \lambda C,$$
$$\phi C = -BE \qquad \alpha(X) = \eta'(BX).$$

Then we have

$$f^2 = -I + \eta\otimes E + \alpha\otimes A,\ \eta f = \lambda\alpha,\ \alpha f = -\lambda\eta,\ fE = -\lambda A,$$
$$fA = \lambda E,\ \eta(E) = 1-\lambda^2,\ \alpha(E) = 0,\ \eta(A) = 0,\ \alpha(A) = 1-\lambda^2\ [1],$$

where I is the identity transformation.

We assume that a global vector field V exists which satisfies $\eta(V)=0$ and $V\neq 0$ on N^{2n}. If we put $\frac{1}{\rho}(-BV+C) = N$ for some nonzero scalar field ρ, then N is an affine normal and we have $\phi N = -BU$ for some vector field U on N^{2n}. Therefore we have $U = \frac{1}{\rho}[fV+E]$, $C = BV + \rho N$.

Since $f^2 = -I+\eta\otimes E+\alpha\otimes A$, $\eta(fU) = \eta\left(\frac{1}{\rho}[f^2V+fE]\right) = 0$. Hence we have

$$\phi BX = BfX + \eta(X)C = BfX + \eta(X)[BV+\rho N] = B(f+\eta\otimes V)X + \rho\eta(X)N,$$

that is, $\phi BX = B\tilde{f}X) + \tilde{\eta}(X)N$, where $\tilde{f} = f+\eta\otimes V$ and $\tilde{\eta} = \rho\eta$.

Thus, $\tilde{f}^2 = -I + \alpha\otimes A + \tilde{\eta}\otimes U + (\eta f)\otimes V$ Therefore we have

$$\begin{aligned}\tilde{f}^4(X) = {}& X - \alpha(X)A - \tilde{\eta}(X)U - \eta(fX)V - \alpha(X)A + \alpha(X)\alpha(A)A + \tilde{\eta}(X)\alpha(U)A \\ & + \eta(fX)\alpha(V)A - \tilde{\eta}(X)U + \alpha(X)\tilde{\eta}(A)U + \tilde{\eta}(X)\tilde{\eta}(U)U + \eta(fX)\alpha(V)A \\ & - \eta(fX)V + \alpha(X)\eta(fA)V + \tilde{\eta}(X)\eta(fU)V + \eta(fX)\eta(fV)V.\end{aligned}$$

From $\tilde{\eta}(X)\alpha(U)A = \alpha(X)\tilde{\eta}(A)U = \eta(fX)\tilde{\eta}(V)\ U = \tilde{\eta}(X)\eta(fU)V = 0$ and $\alpha(X)\alpha(A)A + \eta(fX)\alpha(V)A + \tilde{\eta}(X)\tilde{\eta}(U)U + \alpha(X)\eta(fA)V + \eta(fX)\eta(fV)V = [1-\lambda^2+\alpha(V)\lambda]\tilde{f}^2(X) + [1-\lambda^2+\alpha(V)\lambda]X$, we have

$$\tilde{f}^4(X) + [1+\lambda^2-\lambda\alpha(V)]\tilde{f}^2(X) + (\lambda^2-\lambda\alpha(V)) = 0,\ \text{that is,}\ (\tilde{f}^2+I)[\tilde{f}^2+(\lambda^2-\lambda\alpha(V))I] = 0.$$

Hence we get the following

THEOREM 4. *Let N^{2n} be a hypersurface of almost contact manifold M^{2n+1}. For an arbitrary global nonzero vector field V such that $\eta(V)=0$, if $\lambda^2-\lambda\alpha(V)=1$, then $\tilde{f} = f+\eta\otimes V$ is an almost complex structure.*

Kyungpook University

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