EXPANSIONS OF GENERALISED HYPERGEOMETRIC FUNCTIONS

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1. Introduction. The generalized hypergeometric polynomial [4] has been defined by

$$(1.1)\ F_n(x)=x^{(\delta-1)n}{}_{p+\delta}F_q\left[\begin{matrix}\Delta(\delta,-n),a_1,\cdots,a_p\\ b_1,\cdots,b_q\end{matrix};\lambda x^c\right],$$

where the symbol $\Delta(\delta,-n)$ represents the set of δ-parameters: $-\frac{n}{\delta}$, $\frac{-n+1}{\delta}$, $\cdots$, $\frac{-n+\delta-1}{\delta}$ and δ, n are positive integers. The polynomial is in a generalized form which yields many known polynomials on specializing the parameters.

For ease in writing, we employ the contracted notation

$${}_pF_q(x)={}_pF_q\left(\begin{matrix}a_p\\ b_q\end{matrix}\middle| x\right)=\sum_{r=0}^{\infty}\frac{(a_p)_r\,x^r}{(b_q)_r\,r!}.$$

Thus $(a_p)_r$ is to be interpreted as $\prod_{j=1}^{p}(a_j)_r$ and similarly for $(b_q)_r$. $(a_p,\ e_p)$ denotes $(a_1,e_1),\cdots,(a_p,e_p)$.

In this paper we have established some expansions for Hypergeometric functions. Some particular cases have also been given with proper choice of parameter.

2. The Expansions. Results to be proved are

$$(2.1)\ \sum_{r=0}^{n}(-)^{n-r}\,{}^nC_r\frac{\Gamma\left(\mu+\left(\delta-\frac{1}{2}\right)n-m-r+\frac{1}{2}\right)}{\Gamma\left(\mu+\left(\delta-\frac{1}{2}\right)n-k-r-1\right)}$$

$$\cdot{}_{p+\delta+c}F_{q+2c}\left[\begin{matrix}\Delta(\delta,-n),a_p,\Delta\left(c,-\mu-\left(\delta-\frac{1}{2}\right)n+k+r\right);\\ b_q,\ \Delta\left(c,\frac{1}{2}-\mu-\left(\delta-\frac{1}{2}\right)n-m\right),\ \Delta\left(c,\frac{1}{2}-\mu-\left(\delta-\frac{1}{2}\right)n+m+r\right)\end{matrix}\right.$$

$$\left.;\ \frac{\lambda(-1)^c}{c^c}\right]=\frac{\Gamma\left(m+n-k+\frac{1}{2}\right)\Gamma\left(\mu+(\delta-3/2)n-m+\frac{1}{2}\right)}{\Gamma\left(m-k+\frac{1}{2}\right)\ \Gamma\left(\mu+\left(\delta-\frac{1}{2}\right)n-k+1\right)}\cdot{}_{p+\delta+c}F_{q+2c}$$

$$\left[\begin{matrix}\Delta(\delta,-n),a_p, & \Delta\left(c,-\mu-\left(\delta-\frac{1}{2}\right)n+k\right); & \frac{\lambda(-1)^c}{c^c}\\ b_q,\Delta(c,\frac{1}{2}-\mu-(\delta-1/2)n-m), & \Delta\left(c,\frac{1}{2}-\mu-(\delta-3/2)n+m\right); & \end{matrix}\right],$$

where $R\left(\mu+(\delta-1)n\pm m+\frac{1}{2}\right)>0$; δ, n and c are positive integers.

$$(2.2)\ \sum_{r=0}^{n}(-)^{n-r}{}_nC_r\frac{\Gamma\left(m-k-r+\frac{1}{2}\right)}{\Gamma(\mu+(\delta-1)n-k-r+1)}\cdot{}_{p+q+c}F_{q+2c}$$

$$\cdot\left[\begin{matrix}\Delta(\delta,-n),\ a_p,\ \Delta(c,-\mu-(\delta-1)n+k+r);\\ b_q,\ \Delta\left(c,\frac{1}{2}-\mu-(\delta-1)n-m\right),\ \Delta\left(c,\frac{1}{2}-\mu-(\delta-1)n+m\right);\end{matrix}\ \frac{\lambda(-1)^c}{c^c}\right]$$

$$=\frac{\Gamma\left(m-k-n+\frac{1}{2}\right)\Gamma\left(\mu+\delta n-m+\frac{1}{2}\right)}{\Gamma(\mu+(\delta-1)n-k+1)\Gamma\left(\mu+(\delta-1)n-m+\frac{1}{2}\right)}\cdot{}_{p+q+c}F_{q+2c}$$

$$\cdot\left[\begin{matrix}\Delta(\delta,-n),\ a_p,\ \Delta(c,-\mu-(\delta-1)n+k):\\ b_q,\ \Delta\left(c,\frac{1}{2}-\mu-(\delta-1)n-m\right),\ \Delta\left(c,\frac{1}{2}-\mu-\delta n+m\right);\end{matrix}\ \frac{\lambda(-1)^c}{c^c}\right],$$

where $R\left(\mu+(\delta-1)n\pm m+\frac{1}{2}\right)>0$; δ, n and c are positive integers.

$$(2.3)\ \sum_{r=0}^{n}(-)^{n-r}{}_nC_r\frac{\Gamma\left(\mu+\left(\delta-\frac{1}{2}\right)n-m-r+\frac{1}{2}\right)}{\Gamma\left(\mu+\left(\delta-\frac{1}{2}\right)n-k-r+1\right)}\cdot{}_{p+\delta+2c}F_{q+2c}$$

$$\cdot\left[\begin{matrix}\Delta(\delta,-n),\ a_p,\ \Delta\left(c,\mu+\left(\delta-\frac{1}{2}\right)n+m+\frac{1}{2}\right),\ \Delta\left(c,\mu+(\delta-1)n-m-r+\frac{1}{2}\right);\\ b_q,\ \Delta\left(c,\mu+\left(\delta-\frac{1}{2}\right)n-k-r+1\right)\end{matrix}\ \lambda c^c\right]$$

$$=\frac{\Gamma\left(m+n-k+\frac{1}{2}\right)\Gamma\left(\mu+(\delta-\frac{3}{2})n-m+\frac{1}{2}\right)}{\Gamma\left(m-k+\frac{1}{2}\right)\Gamma\left(\mu+(\delta-\frac{1}{2})n-k+1\right)}{}_{p+\delta+2c}F_{q+2c}$$

$$\cdot\left[\begin{matrix}\Delta(\delta,-n),\ a_p,\ \Delta\left(c,\mu+\left(\delta-\frac{1}{2}\right)n+m+\frac{1}{2}\right),\ \Delta\left(c,\mu+(\delta-\frac{3}{2})n-m+\frac{1}{2}\right);\\ b_q,\ \Delta\left(c,\mu+\left(\delta-\frac{1}{2}\right)n-k+1\right),\end{matrix}\ \lambda c^c\right],$$

where $R\left(\mu+(\delta-1)n\pm m+\frac{1}{2}\right)>0$; δ, n and c are positive integers.

$$(2.4)\ \sum_{r=0}^{n}(-)^{n-r}{}_nC_r\frac{\Gamma\left(m-k-r+\frac{1}{2}\right)}{\Gamma(\mu+(\delta-1)n-k-r+1)}\cdot{}_{p+\delta+2c}F_{q+2c}$$

$$\cdot\left[\begin{matrix}\Delta(\delta,-n),\ a_p, \Delta\left(c,\mu+(\delta-1)n+m+\frac{1}{2}\right),\ \Delta\left(c,\mu+(\delta-1)n-m+\frac{1}{2}\right);\\ b_q,\ \Delta(c,\ \mu+(\delta-1)n-k-r+1);\end{matrix}\ \lambda c^c\right]$$

$$=\frac{\Gamma\left(m-k-n+\frac{1}{2}\right)\Gamma\left(\mu+\delta n-m+\frac{1}{2}\right)}{\Gamma\left(\mu+(\delta-1)n-k+1\right)\Gamma\left(\mu+(\delta-1)n-m+\frac{1}{2}\right)}$$

$$\cdot{}_{p+\delta+2c}F_{q+c}\left[\begin{matrix}\Delta(\delta,-n),a_p,\ \Delta\left(c,\mu+(\delta-1)n+m+\frac{1}{2}\right),\ \Delta\left(c,\mu+\delta n-m+\frac{1}{2}\right);\\ b_q,\ \Delta(c,\mu+(\delta-1)n-k+1):\end{matrix}\ \lambda c^c\right].$$

where $R\left(\mu+(\delta-1)n\pm m+\frac{1}{2}\right)>0;\delta$, n and c are positive integers.

3. Proofs: Since(pathan, 1968; p. 17)

$$(3.1)\ \sum_{r=0}^{n}(-)^{n-r}{}^nC_r\,x^{\frac{n}{2}-\frac{r}{2}}W_{k-\frac{1}{2}r,\ m+\frac{1}{2}r}(x)=\frac{\Gamma\left(m+n-k+\frac{1}{2}\right)}{\Gamma\left(m-k+\frac{1}{2}\right)}\cdot$$

$$\cdot W_{k-\frac{1}{2}n,\ m+\frac{1}{2}n}(x).$$

we have

$$(3.2)\ \int_0^\infty x^\mu f(x)\left\{\sum_{r=0}^{n}(-)^{n-r}{}^nC_r x^{\frac{n}{2}-\frac{r}{2}}W_{k+\frac{1}{2}r,\ m+\frac{1}{2}r}(x)\right\}dx$$

$$=\frac{\Gamma\left(m+n-k+\frac{1}{2}\right)}{\Gamma\left(m-k+\frac{1}{2}\right)}\int_0^\infty x^\mu f(x)\cdot W_{k-\frac{1}{2}n,\ m+\frac{1}{2}n}(x)dx$$

provided that the integrals involved exist. Now if we take

$$(3.3)\ f(x)=x^{(\delta-1)n-1}e^{-\frac{1}{2}x}{}_{p+\delta}F_q\left[\begin{matrix}\Delta(\delta,-n),a_p;\\ b_q;\end{matrix}\ \lambda x^{-c}\right]$$

in (3.2) and evaluate the integrals involved there in with the help of the following results(Shah, [5] 1969; p. 484)

$$(3.4)\ \int_0^\infty x^{\mu-1}e^{-\frac{1}{2}x}W_{k,m}(x)\left\{x^{(\delta-1)n}{}_{p+\delta}F_q\left[\begin{matrix}\Delta(\delta,-n),a_p;\\ b_q;\end{matrix}\ \lambda x^{-c}\right]\right\}dx$$

$$=A\ {}_{p+\delta+c}F_{q+2c}\left[\begin{matrix}\Delta(\delta,-n),a_p,\ \Delta(c,-\mu-(\delta-1)n+k);\\ b_q,\ \Delta\left(c,\frac{1}{2}-\mu-(\delta-1)n-m\right),\ \Delta\left(c,\frac{1}{2}-\mu-(\delta-1)n+m\right);\end{matrix}\ \frac{\lambda(-1)^c}{c^c}\right],$$

where $A=\dfrac{\Gamma\left(\mu+(\delta-1)n+m+\frac{1}{2}\right)\,\Gamma\left(\mu+(\delta-1)n-m+\frac{1}{2}\right)}{\Gamma(\mu+(\delta-1)n-k+1)}$,

$R\left(\mu+(\delta-1)n\pm m+\frac{1}{2}\right)>0$ and δ, n, c are positive integers. We get the recurrence relation(2.1) after some slight changes.

The remaining relations can be proved in a similar manner by using the following results:

$$(3.5)\ \sum_{r=0}^{\infty}\frac{\Gamma\left(m-k-r+\frac{1}{2}\right)}{\Gamma\left(m-k-n+\frac{1}{2}\right)}(-)^{n-r}\,{}^{n}C_{r}W_{k+r,m}(x)=x^{\frac{1}{2}n}W_{k+\frac{1}{2}n,\ m-\frac{1}{2}n}(x)\ \text{and}$$

$$(3.6)\ \int_0^{\infty}x^{\mu-1}e^{-\frac{1}{2}x}W_{k,m}(x)\left\{x^{(\delta-1)}{}_{p+\delta}F_{q}\left[\begin{matrix}\Delta(\delta,-n), & a_p; & \\ b_q; & & \lambda x^c\end{matrix}\right]\right\}dx=A\cdot{}_{p+\delta+2c}F_{q+c}$$

$$\left[\begin{matrix}\Delta(\delta,-n),\ a_p,\ \Delta\left(c,\mu+(\delta-1)n+m+\frac{1}{2}\right),\ \Delta\left(c,\mu+(\delta-1)n-m+\frac{1}{2}\right); \\ b_q,\ \Delta(c,\mu+(\delta-1)n-k+1)\ ; \qquad \lambda c^c\end{matrix}\right],$$

where $R\left(\mu+(\delta-1)n\pm m+\frac{1}{2}\right)>0$ and δ, n, c are positive integers.

Use (3.5) and (3.4) to get (2.2),
Use (3.1) and (3.6) to get (2.3),
Use (3.5) and (3.6) to get (2.4).

4. particular cases: When $\delta=c=\lambda=1$ in (2.3):

(a) with $a_1=n+\alpha+\beta+1$, $b_1=1+\alpha$, $b_2=\frac{1}{2}$, we obtain

$$(4.1)\ \sum_{r=0}^{n}(-)^{n-r}\,{}^{n}C_r\frac{\Gamma\left(\mu+\frac{n}{2}-m-r+\frac{1}{2}\right)}{\Gamma\left(\mu+\frac{n}{2}-k-r+1\right)}f_n^{(\alpha,\beta)}$$

$$\times\left(\begin{matrix}a_2,\cdots a_p,\ \mu+\frac{n}{2}+m+\frac{1}{2},\ \mu+\frac{n}{2}-m-r-\frac{1}{2} \\ b_3,\cdots b_q,\ \mu+\frac{n}{2}-k-r+1;\ 1\end{matrix}\right)$$

$$=\frac{\Gamma\left(m+n-k+\frac{1}{2}\right)\,\Gamma\left(\mu-m+\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(m-k+\frac{1}{2}\right)\,\Gamma\left(\mu+\frac{n}{2}-k+1\right)}f_n^{(\alpha,\beta)}$$

$$\times\begin{pmatrix} a_2, \cdots, a_p, & \mu+m+\frac{n}{2}+\frac{1}{2}, & \mu-m-\frac{n}{2}-\frac{1}{2} \\ b_3, \cdots, b_q, & \mu-k+\frac{n}{2}+1; 1 & \end{pmatrix},$$

where $R\left(\mu\pm m+\frac{n}{2}+\frac{1}{2}\right)>0$ and $f_n^{(\alpha,\beta)}$ is a generalized Sister celine's polynomials [1].

(4.2) With $p=0$, $q=1$, $b_1=1+\alpha$, $m=\alpha+\beta$, $\mu=\frac{n}{2}+\frac{1}{2}$, we obtain

$$\sum_{r=0}^{n}(-)^{n-r}{}^nC_r\,\frac{\Gamma(n-\alpha-\beta-r+1)}{\Gamma(n-k-r+3/2)}H_n^{\alpha,\beta}\left(n-\alpha-\beta-\gamma+1,\ n-k-r+\frac{3}{2};\ 1\right)$$

$$=\frac{\Gamma\left(n+\alpha+\beta-k+\frac{1}{2}\right)\,\Gamma(1-\alpha-\beta-n)}{\Gamma\left(\alpha+\beta-k+\frac{1}{2}\right)\,\Gamma\left(n-k+\frac{3}{2}\right)}H_n^{\alpha,\beta}\left(1-\alpha-\beta,\ n-k+\frac{3}{2};1\right),$$

where $H_n^{\alpha,\beta}$ is a generalized Rice's polynomial [2].

(4.3) With $p=0$, $q=1$, $m=-\mu-3n/2$, $b_1=\mu$, $k=-1/2$; we get

$$\sum_{r=0}^{n}\ (-)^{n-r}\,{}^nC_r\,\frac{\Gamma\left(2\mu+2n-r+\frac{1}{2}\right)}{(\Gamma\mu+3n/2-r+3/2)\ (-\mu-3n/2-1/2+r)_{2n}}$$

$$\cdot R_{2n}\left(-\mu-\frac{3n}{2}-\frac{1}{2}+r,\mu:1\right)=\frac{\Gamma(1-\mu-n/2)\ \Gamma\left(2\mu+n+\frac{1}{2}\right)}{\Gamma(1-\mu-3n/2)\ \Gamma(\mu+n/2+3/2)}\,{}_3F_2$$

$$\left[\begin{matrix} -n, & -n+\frac{1}{2}, & 2\mu+n+\frac{1}{2} \\ \mu, & \mu+\frac{n}{2}+\frac{3}{2} & \end{matrix};1\right],$$

where $R_n(\beta,r;x)$ is a Bedient's polynomial [3, 297(1)]

(b) when $c=2$, $\delta=1$, $\lambda=4$, $a_1=n+\alpha+\beta+1$, $b_1=1+\alpha$, $b_2=\frac{1}{2}$ in (2.1), we get

(4.4) $$\sum_{r=0}^{n}(-1)^{n-r}\frac{\Gamma(\mu-m-r+n/2+1/2)}{\Gamma(\mu-k-r+n/2-1)}f_n^{(\alpha,\beta)}$$

$$\times\left[\begin{matrix} a_2, \cdots, a_p, -\mu-\frac{n}{2}+k+r;\ 1 & \\ b_3, \cdots, b_q, 1/2-\mu-n/2-m, & 1/2-n/2-\mu+m+3 \end{matrix}\right]$$

$$=\frac{\Gamma(m+n-k+1/2)\ \Gamma(\mu-m+n/2+1/2)}{\Gamma(m-k+1/2)\ \Gamma(\mu-k+n/2+1)}\ f_n^{(\alpha,\beta)}$$

$$\times\left[\begin{matrix} a_2, \cdots, a_p, k-\mu-n/2; \\ b_3, \cdots, b_q, 1/2-n/2-\mu-m, \ 1/2+n/2-\mu+m; \end{matrix} \ 1\right],$$

$R(m+n-k+1/2)>0$, $R(\mu-m+n/2)\geqq 0$.

(4.5) When $\delta=c=2$, $p=1, q=2, a_1=r-\beta$, $b_1=r-\beta$, $b_2=1-\beta-n$, $\lambda=4$ in (2.1), We get

$$\sum_{r=0}^{n}(-)^{n-r}\, {}^nC_r \frac{\Gamma(\mu+3n/2-m-r+1/2)}{\Gamma(\mu+3n/2-k-r+1)}\, {}_5F_6$$

$$\times\left[\begin{matrix} \Delta(2,-n),\ r-\beta,\ \Delta(2,-\mu-3n/2-m-r+1/2 \\ r, 1-\beta-n,\ \Delta(2, 1/2-\mu-3n/2\pm m+r)^* ; 1 \end{matrix}\right]$$

$$=\frac{\Gamma\left(m+n-k+\frac{1}{2}\right)\Gamma(\mu+n/2-m+1/2)}{\Gamma\left(m-k+\frac{1}{2}\right)\Gamma(\mu+3n/2-k+1)}\, {}_5F_6$$

$$\times\left[\begin{matrix} \Delta(2,-n),\ r-\beta,\ \Delta(2,-\mu-3n/2-k) \\ r, 1-\beta-n,\ \Delta(2, 1/2-\mu-3n/2\pm m) \end{matrix} ; 1\right],$$

where $R(m+n-k)\geqq 0$, $R(\mu+n/2-m+1/2)>0$ and $^*\Delta(2,\ 1/2-\mu-3n/2\pm m+r)$ $=\Delta(2,\ \frac{1}{2}-\mu-3n/2+m+r),\ \Delta(2, 1/2-\mu-3n/2-m+r)$.

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