# ON MEIJER-LAPLACE TRANSFORM OF TWO VARIABLES

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## 1. Introduction

The Meijer-Laplace transform is defined [1, p. 57] by means of the integral equation in the form

$$(1,1)\quad F(p)=p\int_0^\infty G^{m+1,\,0}_{m,\,m+1}\left(px \,\middle|\, \begin{matrix} a_1+b_1, \cdots\cdots, & a_m+b_m \\ a_1, \cdots\cdots\cdots\cdots, & a_{m+1} \end{matrix}\right) f(x)dx,\quad R(p)>0$$

and denote it symbolically as

$$F(p)=G[f(x);\ a_{m+1}\ b_m].$$

The author has introduced the Meijer-Laplace transform of two variables [6, 7] as

$$(1,2)\quad F(p,q)=pq\int_0^\infty\int_0^\infty G^{m+1,\,0}_{m,\,m+1}\left(px \,\middle|\, \begin{matrix} a_1+b_1, \cdots, & a_m+b_m \\ a_1, \cdots\cdots\cdots, & a_{m+1} \end{matrix}\right) G^{n+1,\,0}_{n,\,n+1}\left(qy \,\middle|\, \begin{matrix} c_1+d_1, \cdots, c_n+d_n \\ c_1, \cdots\cdots\cdots, c_{n+1} \end{matrix}\right)$$
$$\times f(x,y)dxdy,\ R(p,q)>0$$

which is the generalisation of well known Laplace transform of two variables [5]

$$(1,3)\quad F(p,q)=pq\int_0^\infty\int_0^\infty e^{-px-qy} f(x,\ y)dxdy,\ R(p,q)>0.$$

When $b_j=0,\ j=1, 2, \cdots\cdots,\ m-1;\ d_k=0,\ k=1, 2, \cdots\cdots,\ n-1$ and

(a) $a_{m+1}=b_m=c_{n+1}=d_n=0$, the integral equation (1, 2) reduces to (1, 3);

(b) $b_m=a_{m+1}=-m_1-k_1,\ a_m=m_1-k_1;\ c_{n+1}=d_n=-m_2-k_2,\ c_n=m_2-k_2$, the equation (1, 2) reduces to

$$(1,4)\quad F(p,q)=pq\int_0^\infty\int_0^\infty e^{-\frac{1}{2}px-\frac{1}{2}qy}(px)^{-k_1-\frac{1}{2}}(qy)^{-k_2-\frac{1}{2}}W_{k_1+\frac{1}{2},\ m_1}(px)$$
$$\times W_{k_2+\frac{1}{2},\ m_2}(qy)\ f(x,\ y)dxdy,\ R(p,q)>0$$

and is known as Meijer transform of two variables [9, p. 83];

(c) $b_m=2m_1,\ a_m=\frac{1}{2}-m_1-k_1,\ a_{m+1}=0;\ d_n=2m_2,\ c_n=\frac{1}{2}-m_2-k_2,\ c_{n+1}=0,$ integral equation (1, 2) reduces to

$$(1,5)\quad F(p,q)=pq\int_0^\infty\int_0^\infty e^{-\frac{1}{2}px-\frac{1}{2}qy}(px)^{m_1-\frac{1}{2}}(qy)^{m_2-\frac{1}{2}}W_{k_1,\ m_1}(px)\ W_{k_2,\ m_2}(qy)$$
$$\times f(x,y)dxdy,\ R(p,q)>0$$

which we shall call as Varma transform [10].

We shall denote (1, 2) symbolically as

$$F(p,q)=G\left[f(x,y);\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\right],$$

and (1, 3), (1, 4) and (1, 5) as usual shall be denoted as

$$F(p,q)\doteqdot f(x,y),\quad F(p,q)\xleftarrow[m_1,\ m_2]{k_1+\frac{1}{2},\ k_2+\frac{1}{2}} f(x,y) \text{ and } F(p,q)\xleftarrow[m_1,\ m_2]{k_1,\ k_2} f(x,y)$$

respectively.

The object of the present paper is to establish some rules and obtain the images of some of the functions in Meijer-Laplace transform. Some differential and integral properties of the transform (1, 2) also have been discussed. Particular cases of the theorems give rise to results obtained earlier by Bose [2] Mehra [9] and Poli and Delerue [11].

In what follows, we have used the symbol $(a_k)$ to denote the set of parameters $a_1, \cdots\cdots, a_k$ throughout this paper.

**2.**

THEOREM 1. *If* $f(x,y)=f_1(x)\cdot f_2(y)$, *and if*

$$F_1(p)=G[f_1(x);\ a_{m+1}, b_m],$$
$$F_2(q)=G[f_2(y);\ c_{n+1}, d_n], \text{ then}$$

$$(2,1)\quad F(p,q)=F_1(p)\cdot F_2(q)=G\left[f(x,y);\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\right],$$

*provided the integrals are absolutely convergent.*

EXAMPLE 1. Taking $f(x,y)=x^{v_1}y^{v_2}e^{-ux-wy}$, and using [4, p. 419, (5)] and (2, 1), we get

$$(2,2)\quad \frac{pq}{u^{v_1+1}\,w^{v_2+1}}G^{m+1,\ 1}_{m+1,m+1}\left(\frac{p}{u}\,\middle|\,\begin{matrix} -v_1, (a_m+b_m) \\ (a_{m+1}) \end{matrix}\right)G^{n+1,\ 1}_{n+1,n+1}\left(\frac{q}{w}\,\middle|\,\begin{matrix} -v_2,\ (c_n+d_n) \\ (c_{n+1}) \end{matrix}\right)$$
$$=G\left[x^{v_1}y^{v_2}e^{-ux-wy};\ \begin{matrix} a_{m+1}, b_m \\ c_{n+1}, d_n \end{matrix}\right],$$

provided $R(a_j+v_1+1)>0$, $j=1, 2, \cdots\cdots, m+1$ ; $R(c_k+v_2+1)>0$, $k=1, 2, \cdots\cdots, n+1$ ; $|\arg p|<\frac{\pi}{2}$, $|\arg q|<\frac{\pi}{2}$, $|\arg u|<\frac{\pi}{2}$, $|\arg w|<\frac{\pi}{2}$ and $R(p,\ q)>0$.

In the above result, putting $b_j=0, j=1, 2, \cdots\cdots, m$; $a_{m+1}=0$, $d_k=0$, $k=1, 2, \cdots, n$:

$c_{n+1}=0$, $v_1=\nu$, $v_2=\nu_1$, $u=a$, $w=a_1$ and simplifying, we obtain a known result [9, p. 87]

$$(2,3)\quad \frac{\Gamma(\nu+1)\,\Gamma(\nu_1+1)}{(a+p)^{\nu+1}\,(a_1+q)^{\nu_1+1}} \doteqdot x^{\nu}y^{\nu_1}e^{-ax-a_1y},$$ where $R(p+a,q+a)>0$ and $R(\nu,\nu_1)>-1$. Further, with $a=a_1=0$, we have [3, p. 100, (1.3)]

$$(2,4)\quad \frac{\Gamma(\nu+1)\,\Gamma(\nu_1+1)}{p^{\nu}q^{\nu_1}} \doteqdot x^{\nu}y^{\nu_1},$$

provided $R(p,q)>0$ and $R(\nu,\nu_1)>-1$.

**3.**

THEOREM 2. *If* $F(p,q)=G\left[f(x,\ y);\ \begin{matrix}a_{m+1}, b_m\\ c_{n+1}, d_n\end{matrix}\right]$,

*then*

$$(3,1)\quad F\left(\frac{p}{a},\ \frac{q}{b}\right)=G\left[f(ax,\ by);\ \begin{matrix}a_{m+1}, b_m\\ c_{n+1}, d_n\end{matrix}\right],$$

*provided the integrals are absolutely convergent and a and b are positive integers.*

EXAMPLE 1. Taking $f(x,\ y)=x^{r_1}y^{r_2}$ and using [4, p. 418, (3)], we have

$$(3,2)\quad F(p,q)=\frac{\prod_{j=1}^{m+1}\Gamma(a_j+r_1+1)\prod_{j=1}^{n+1}\Gamma(c_j+r_2+1)}{\prod_{j=1}^{m}\Gamma(a_j+b_j+r_1+1)\prod_{j=1}^{n}\Gamma(c_j+d_j+r_2+1)}\,p^{-r_1}q^{-r_2},$$

provided $R(a_j+r_1+1)>0$, $j=1,2,\cdots,m+1$, $|\arg p|<\frac{\pi}{2}$, $R(c_k+r_2+1)>0$, $k=1$, $2,\cdots\cdots$, $n+1$, $|\arg q|<\frac{\pi}{2}$ and $R(p,q)>0$.

Hence, from (3.1), we obtain

$$(3,3)\quad \frac{\prod_{j=1}^{m+1}\Gamma(a_j+r_1+1)\prod_{j=1}^{n+1}\Gamma(c_j+r_2+1)}{\prod_{j=1}^{m}\Gamma(a_j+b_j+r_1+1)\prod_{j=1}^{n}\Gamma(c_j+d_j+r_2+1)}\left(\frac{p}{a}\right)^{-r_1}\left(\frac{q}{b}\right)^{-r_2}$$
$$=G\left[(ax)^{r_1}(by)^{r_2};\begin{matrix}a_{m+1},\ b_m\\ c_{n+1},\ d_n\end{matrix}\right],$$

provided the conditions given in (3,2) are satisfied and $a$ and $b$ are positive integers.

Setting $b_j=0, j=1,\ 2,\ \cdots\cdots,\ m$; $a_{m+1}=0$, $d_k=0$, $k=1,\ 2,\ \cdots\cdots,\ n$, $c_{n+1}=0$, $r_1=n$ and $r_2=m$ in (3.3), we obtain [11, p. 21]

$$\frac{a^n}{p^n}\cdot\frac{b^m}{q^m} \doteqdot \frac{(ax)^n\,(by)^m}{\Gamma(n+1)\Gamma(m+1)}\,.$$

4.

THEOREM 3. *If*

$$F_1(p,q)=G\left[f_1(x,y);\ \begin{matrix}a_{m+1}, & b_m\\ c_{n+1}, & d_n\end{matrix}\right],$$

$$F_2(p,q)=G\left[f_2(x,y);\ \begin{matrix}a_{m+1}, & b_m\\ c_{n+1}, & d_n\end{matrix}\right]$$

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$$F_s(p,q)=G\left[f_s(x,y);\ \begin{matrix}a_{m+1}, & b_m\\ c_{n+1}, & d_n\end{matrix}\right],$$

*then*

$$\sum_{r=1}^{s}F_r(p,q)=G\left[\sum_{r=1}^{s}f_r(x,y);\ \begin{matrix}a_{m+1}, & b_m\\ c_{n+1}, & d_n\end{matrix}\right], \tag{4, 1}$$

*provided* $R(a_j+u_r+1)>0$, $j=1, 2, \cdots\cdots, m+1$ ; $r=1, 2, \cdots\cdots, s$, $|\arg p|<\frac{\pi}{2}$,

$R(c_k+v_r+1)>0$, $k=1, 2, \cdots\cdots, n+1$ ; $r=1, 2, \cdots\cdots, s$, $|\arg q|<\frac{\pi}{2}$,

$R(p,q)>0$, *where* $f_r(x,y)=0(x^{u_r}, y^{v_r})$ *for small* $x$ *and* $y$ *and*

$$G^{m+1,\ 0}_{m,\ m+1}\left(p_0x\,\middle|\,\begin{matrix}(a_m+b_m)\\(a_{m+1}),\end{matrix}\right),\ G^{n+1,\ 0}_{n,\ n+1}\left(q_0y\,\middle|\,\begin{matrix}(c_n+d_n)\\(c_{n+1})\end{matrix}\right)$$

*are bounded for* $R(p)\geqslant p_0>0$ *and* $R(q)\geqslant q_0>0$ *respectively.*

The above theorem also holds when $s$ tends to infinity, provided

(i) $\sum_{r=1}^{\infty} f_r(x,y)$ is absolutely convergent for $x>0$ and $y>0$ ;

(ii) $\sum_{r=1}^{\infty} F_r(p,q)$ is absolutely convergent; and

(iii) the integrals involved are absolutely convergent for $R(a_j+u_r+1)>0$, $R(c_k+v_r+1)>0$, $j=1,2, \cdots\cdots, m+1$; $k=1,2, \cdots\cdots, n+1$, $|\arg p|<\frac{\pi}{2}$, $|\arg q|<\frac{\pi}{2}$ and $R(p,q)>0$, where $f(x,y)=0(x^{u_r}, y^{v_r})$ for small $x$ and $y$, and

$$G^{m+1,\ 0}_{m,\ m+1}\left(p_0x\,\middle|\,\begin{matrix}(a_m+b_m)\\(a_{m+1})\end{matrix}\right)f_r(x,y),\ G^{n+1,\ 0}_{n,\ n+1}\left(q_0y\,\middle|\,\begin{matrix}(c_n+d_n)\\(c_{n+1})\end{matrix}\right)f_r(x,y)$$

are bounded for $x>0$, $R(p)\geqslant p_0>0$ and $y>0$, $R(q)\geqslant q_0>0$ respectively.

EXAMPLE 1. Taking $f(x,y)=(x+y)^k=\sum_{r=0}^{k} {}^kC_r\, x^{k-r}y^r$, where $k$ is a positive integer. On applying [4, p. 418, (3)] and (4.1), we have

$$\sum_{r=0}^{k} {}^kC_r \frac{\prod_{j=1}^{m+1}\Gamma(a_j+k-r+1)\prod_{j=1}^{n+1}\Gamma(c_j+r+1)}{\prod_{j=1}^{m}\Gamma(a_j+b_j+k-r+1)\prod_{j=1}^{n}\Gamma(c_j+d_j+r+1)} p^{-k+r}q^{-r} \tag{4,2}$$

$$=G\left[(x+y)^k;\begin{matrix}a_{m+1}, b_m\\ c_{n+1}, d_n\end{matrix}\right],$$

provided $R(a_j+1)>0$, $j=1, 2, \cdots\cdots, m+1$ ; $R(c_k+1)>0$, $k=1, 2, \cdots\cdots, n+1$, $|\arg p|<\frac{\pi}{2}$, $|\arg q|<\frac{\pi}{2}$ and $R(p, q)>0$.

In (4,2), setting $b_j=0$, $j=1, 2, \cdots\cdots, m$; $a_{m+1}=0$ ; $d_k=0$, $k=1, 2, \cdots\cdots, n$, $c_{n+1}=0$, $k=n$ we get a result [8, p. 62]

$$\frac{n!}{(pq)^n}\,\frac{q^{n+1}-p^{n+1}}{q-p} \doteqdot (x+y)^n.$$

EXAMPLE 2. Taking $f(x,y)=(xy)^{\frac{1}{2}N}J_N\left(2(xy)^{\frac{1}{2}}\right)$,

$$=\sum_{r=0}^{\infty}\frac{(-1)^r(xy)^{N+r}}{r!\,\Gamma(N+r+1)}$$

then

$$\left\{\begin{aligned}
&\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(N+r+1)}\cdot\frac{\prod_{j=1}^{m+1}\Gamma(a_j+N+r+1)\prod_{j=1}^{n+1}\Gamma(c_j+N+r+1)}{\prod_{j=1}^{m}\Gamma(a_j+b_j+N+r+1)\prod_{j=1}^{n}\Gamma(c_j+d_j+N+r+1)}\\
&\times(pq)^{-N-r}=\frac{1}{(pq)^N}\\
&\times\frac{\prod_{j=1}^{m+1}\Gamma(a_j+N+1)\prod_{j=1}^{n+1}\Gamma(c_j+N+1)}{\Gamma(N+1)\prod_{j=1}^{m}\Gamma(a_j+b_j+N+1)\prod_{j=1}^{n}\Gamma(c_j+d_j+N+1)}\\
&\times{}_{m+n+2}F_{m+n+1}\left[\begin{matrix}(a_{m+1}+N+1),\ (c_{n+1}+N+1)\\ N+1,\ (a_m+b_m+N+1),\ (c_n+d_n+N+1)\end{matrix};-\frac{1}{pq}\right]\\
&=G\left[(xy)^{\frac{1}{2}N}J_N(2(xy)^{\frac{1}{2}})\,;\begin{matrix}a_{m+1}, & b_m\\ c_{n+1}, & d_n\end{matrix}\right],
\end{aligned}\right. \tag{4,3}$$

provided $R(N+a_j+1)>0$, $j=1, 2, \cdots, m+1$ ; $R(N+c_k+1)>0$, $c_k=1, 2, \cdots, n+1$, $R\left(\frac{1}{2}N+\frac{3}{4}\right)<0$, $|pq|>1$, $|\arg p|<\frac{\pi}{2}$, $|\arg q|<\frac{\pi}{2}$ and $R(p, q)>0$.

Setting $b_j=0, j=1, 2, \cdots\cdots, m$; $d_k=0$, $k=1, 2, \cdots\cdots, n$, $a_{m+1}=c_{n+1}=0$, we get a result due to Bose [2, p. 27]

$$(4, 4)\qquad \frac{\Gamma(N+1)pq}{(1+pq)^{N+1}} \doteqdot (xy)^{\frac{1}{2}N}\, J_N(2(xy)^{\frac{1}{2}}).$$

5.

THEOREM 4. *If*

$$F_1(p, q)=G\left[f_1(x, y):\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\right],$$

*and*

$$F_2(p, q)=G\left[f_2(x, y):\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\right],$$

*then*

$$(5, 1)\qquad \int_0^\infty\int_0^\infty F_1(u, v)f_2(u, v)\ \frac{dudv}{uv}=\int_0^\infty\int_0^\infty F_2(u, v)f_1(u, v)\ \frac{dudv}{uv},$$

***provided the integrals are absolutely convergent.***

PROOF. From the definition (1, 2), we have

$$(5, 2)\quad F_1(u, v)=uv\int_0^\infty\int_0^\infty G^{m+1,\ 0}_{m,\ m+1}\left(ux \,\middle|\, \begin{matrix}(a_m+b_m)\\(a_{m+1})\end{matrix}\right) G^{n+1,\ 0}_{n,\ n+1}\left(vy \,\middle|\, \begin{matrix}(c_n+d_n)\\(c_{n+1})\end{matrix}\right) f_1(x, y)dxdy,$$

and

$$(5, 3)\quad F_2(u, v)=uv\int_0^\infty\int_0^\infty G^{m+1,\ 0}_{m,\ m+1}\left(ux \,\middle|\, \begin{matrix}(a_m+b_m)\\(a_{m+1})\end{matrix}\right) G^{n+1,\ 0}_{n,\ n+1}\left(vy \,\middle|\, \begin{matrix}(c_n+d_n)\\(c_{n+1})\end{matrix}\right) f_2(x, y)dxdy.$$

Now

$$\int_0^\infty\int_0^\infty F_1(u, v)f_2(u, v)\ \frac{dudv}{uv}$$

$$=\int_0^\infty\int_0^\infty f_2(u, v)dudv\ \left[\int_0^\infty\int_0^\infty G^{m+1,\ 0}_{m,\ m+1}\left(ux \,\middle|\, \begin{matrix}(a_m+b_m)\\(a_{m+1})\end{matrix}\right) G^{n+1,\ 0}_{n,\ n+1}\left(vy \,\middle|\, \begin{matrix}(c_n+d_n)\\(c_{n+1})\end{matrix}\right)\right.$$

$$\left.\times f_1(x, y)\, dxdy\right]=\int_0^\infty\int_0^\infty f_1(x, y)\, dxdy\left[\int_0^\infty\int_0^\infty G^{m+1,\ 0}_{m,\ m+1}\left(ux \,\middle|\, \begin{matrix}(a_m+b_m)\\(a_{m+1})\end{matrix}\right)\right.$$

$$\left.\times G^{n+1,\ 0}_{n,\ n+1}\left(vy \,\middle|\, \begin{matrix}(c_n+d_n)\\(c_{n+1})\end{matrix}\right) f_2(u, v)dudv\right]=\int_0^\infty\int_0^\infty f_1(x, y)F_2(x, y)\frac{dxdy}{xy}$$

which follows from (5, 2).

The change in the order of integration is justified as the integrals involved are absolutely convergent. Regarding this change, we observe that the conditions for absolute convergence of (5, 2) are $R(a_j+k_1+1)>0$, $j=1,2, \cdots\cdots, m+1$ : $R(c_i+k_2+1)>0$, $i=1,2, \cdots\cdots, n+1$ : where $f_1(x,y)=0(x^{k_1}, y^{k_2})$, for small $x$ and $y$, and $f_1(x,y)=0\ \{\exp(-x^{s_1}), \exp(-y^{s_2})\}$, $R(s_1,s_2)>0$ for large $x$ and $y$.

Similarly the integral (5, 3) will be absolutely convergent if $R(a_j+k_3+1)>0$, $j=1,2, \cdots\cdots, m+1$ ; $R(c_i+k_4+1)>0$, $i=1,2, \cdots\cdots, n+1$, where $f_2(x,y)=0(x^{k_3}, y^{k_4})$ for small $x$ and $y$, and

$$f_2(x,y)=0\{\exp(-x^{s_3}), \exp(-y^{s_4})\}, \ R(s_3,s_4)>0, \text{ for large } x \text{ and } y.$$

**6.**

THEOREM 6. *If*

$$F\left(\frac{p}{a}, \frac{q}{b}\right)=G\left[f(ax,by); \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\right], \tag{6, 1}$$

*then*

$$\int_p^\infty\int_q^\infty F(u,v)g\left(\frac{p}{u}, \frac{q}{v}\right)\frac{dudv}{uv} \tag{6, 2}$$

$$=G\left[\int_0^x\int_0^y f(s,t)g\left(\frac{s}{x}, \frac{t}{y}\right)\frac{dsdt}{st}; \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\right],$$

$$\int_0^p\int_0^q F(u,v)g\left(\frac{p}{u}, \frac{q}{v}\right)\frac{dudv}{uv} \tag{6, 3}$$

$$=G\left[\int_x^\infty\int_y^\infty f(s,t)g\left(\frac{s}{x}, \frac{t}{y}\right)\frac{dsdt}{st}; \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\right],$$

*and*

$$\int_0^\infty\int_0^\infty F(u,v)g\left(\frac{p}{u}, \frac{q}{v}\right)\frac{dudv}{uv} \tag{6, 4}$$

$$=G\left[\int_0^\infty\int_0^\infty f(s,t)g\left(\frac{s}{x}, \frac{t}{y}\right)\frac{dsdt}{st}; \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\right],$$

*provided the integrals involved are absolutely convergent.*

PROOF. Multiplying both the sides of (6, 1) by $\frac{g(a,b)}{ab}$ and integrating with respect to $a$ and $b$ from 0 to 1, we get

$$(6,5)\qquad \int_0^1\int_0^1 F\Big(\frac{p}{a},\ \frac{q}{b}\Big)g(a,b)\frac{dadb}{ab}$$
$$=G\Big[\int_0^1\int_0^1 f(ax,by)\,g\,(a,b)\ \frac{dadb}{ab}\ ;\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\Big].$$

Substituting in the left hand side $a=\frac{p}{u}$ and $b=\frac{q}{v}$, and on the right hand side $ax=s$ and $by=t$, we obtain

$$\int_p^\infty\int_q^\infty F(u,\ v)g\left(\frac{p}{u},\ \frac{q}{v}\right)\ \frac{dudv}{uv}=G\Big[\int_0^x\int_0^y f(s,t)g\Big(\frac{s}{x},\ \frac{t}{y}\Big)\frac{dsdt}{st}\ ;\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\Big].$$

Thus the result (6, 2) is proved.

To prove (6, 3), multiplying both the sides of (6, 1) by $\frac{g(a,b)}{ab}$, integrating with respect to $a$ and $b$ from 1 to $\infty$, and substituting as in (6, 2), the result (6, 3) is obtaind.

Instead of integrating from 1 to $\infty$, if we integrate in the above from 0 to $\infty$, and changing the variables suitably the result (6, 4) is obtained.

COROLLARY A. *If we take* $g(a,b)=a^{-M}b^{-N}$, *then from* (6, 2), (6, 3) *and* (6, 4), *we have*

$$(6,6)\quad \frac{1}{p^M q^N}\int_p^\infty\int_q^\infty F(u,v)u^{M-1}\,v^{N-1}\ dudv$$
$$=G\Big[x^M y^N\int_0^x\int_0^y f(s,t)\ \frac{dsdt}{s^{M+1}\ t^{N+1}};\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\Big]$$

$$(6,7)\quad \frac{1}{p^M q^N}\int_0^p\int_0^q F(u,v)\,u^{M-1}\,v^{N-1}\,dudv$$
$$=G\Big[x^M y^N\int_x^\infty\int_y^\infty f(s,t)\ \frac{dsdt}{s^{M+1}\ t^{N+1}};\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\Big]$$

*and*

$$(6,8)\quad \frac{1}{p^M q^N}\int_0^\infty\int_0^\infty F(u,v)\,u^{M-1}v^{N-1}\ dudv$$
$$=G\left[x^M y^N\int_0^\infty\int_0^\infty f(s,t)\ \frac{dsdt}{s^{M+1}\,t^{N+1}}\ ;\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\right]$$

*provided the integrals involved are absolutely convergent.*

COROLLARY B. *If we take*

$$g(a,b)=\frac{ab}{(a^2+1)\ (b^2+1)},$$

*in* (6, 4), *we get*

$$(6,9)\quad pq\int_0^\infty\int_0^\infty \frac{F(u,v)}{(u^2+p^2)(v^2+q^2)}\,dudv$$

$$=G\left[xy\int_0^\infty\int_0^\infty \frac{f(s,t)}{(s^2+x^2)(t^2+y^2)}\,dsdt;\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\right],$$

*provided both sides exist.*

Setting $b_j=0, j=1,2,\ \cdots\cdots,\ m;\ d_k=0,\ k=1,2,\ \cdots\cdots,\ n,\ a_{m+1}=0,\ c_{n+1}=0$, in (6,2) to (6,9), we obtain the results due to Delerue [11, p. 27—28].

**7. Differential property.**

Differentiating the equality (3,1) with respect to $a$ and $b$ partially and then putting $a$ and $b$ equal to unity, we obtain

$$(7,1)\quad -p\frac{\partial}{\partial p}[F(p,q)]=G\left[x\frac{\partial}{\partial x}f(x,y);\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\right],$$

$$(7,2)\quad -q\frac{\partial}{\partial q}[F(p,q)]=G\left[y\frac{\partial}{\partial y}f(x,y);\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\right],$$

and

$$(7,3)\quad pq\frac{\partial^2}{\partial p\partial q}[F(p,q)]=G\left[xy\frac{\partial^2}{\partial x\partial y}f(x,y);\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\right],$$

provided both the sides exist and are continuous.

**8. Integral property.**

If we divide the equality (3,1) by $a$ and $b$ and then integrate it with respect to $a$ and $b$ respectively between the limits 0 to 1, we obtain

$$(8,1)\quad \int_p^\infty F(a,q)\frac{da}{a}=G\left[\int_0^x f(a,y)\frac{da}{a};\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\right],$$

$$(8,2)\quad \int_q^\infty F(p,b)\frac{db}{b}=G\left[\int_0^y f(x,b)\frac{db}{b};\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\right],$$

and

$$(8,3)\quad \int_p^\infty\int_q^\infty F(a,b)\frac{dadb}{ab}=G\left[\int_0^x\int_0^y f(a,b)\frac{dadb}{ab};\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\right],$$

provided the integrals are convergent.

Instead of taking the limits from 0 to 1, if we take the limits from 0 to $\infty$, we have

$$(8,4)\quad \int_0^\infty F(a,q)\frac{da}{a}=G\Big[\int_0^\infty f(a,y)\frac{da}{a};\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\Big],$$

$$(8,5)\quad \int_0^\infty F(p,b)\frac{db}{b}=G\Big[\int_0^\infty f(x,b)\frac{db}{b};\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\Big],$$

and

$$(8,6)\quad \int_0^\infty\int_0^\infty F(a,b)\frac{dadb}{ab}=G\Big[\int_0^\infty\int_0^\infty f(a,b)\frac{dadb}{ab};\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\Big],$$

provided the integrals are convergent.

Substracting (8, 1), (8, 2) and (8, 3) from (8, 4), (8, 5) and (8, 6) respectively, we get

$$(8,7)\quad \int_0^p F(a,q)\frac{da}{a}=G\Big[\int_x^\infty f(a,y)\frac{da}{a};\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\Big],$$

$$(8,8)\quad \int_0^q F(p,b)\frac{db}{b}=G\Big[\int_y^\infty f(x,b)\frac{db}{b};\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\Big],$$

and

$$(8,9)\quad \int_0^p\int_0^q F(a,b)\frac{dadb}{ab}=G\Big[\int_x^\infty\int_y^\infty f(a,b)\frac{dadb}{ab};\ \begin{matrix} a_{m+1}, & b_m \\ c_{n+1}, & d_n \end{matrix}\Big],$$

provided the integrals are convergent.

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