A TABLE OF ELEMENTARY EXTENDED GROUP

By Sang Moon Kim

Introduction

Let E be a group of transformations $\frac{az+b}{cz+d}$ or $\frac{a\bar{z}+b}{c\bar{z}+d}$. Then E contains a normal subgroup G of index two which consists of all the Möbius transformations contained in E. If there is a sequence of distinct elements $\{A_n\}$, $A_n \in E$ (or $A_n \in G$), and a point t contained in the extended complex plane C such that $\lim_{n\to\infty} A_n(z) = t$, then we call t is a *limit point* of the group E (or G).

Let R be the set of non-limit point of E (or G) then we call R the *region of discontinuity* of E (or G). If R is not empty then we call E (or G) an *extended group* (or *Kleinian group*).

An extended group (or Kleinian group) with at most two limit points is called an *elementary extended group* (or *elementary Kleinian group*).

Ford [1], gives a complete table of an elementary Kleinian group, this paper gives a table of elementary extended group.

1. Groups with one limit point

Let G be a Kleinian group with one limit point. Then it is conjugate to one of the following groups which we represent in terms of its generators.

(1) $z+1$

(2) $z+1,\ -z$

(3) $z+1,\ z+p$

(4) $z+1,\ z+p,\ -z$

(5) $z+1,\ z+i,\ iz$

(6) $z+1,\ z+\exp\left(\frac{2}{3}\pi i\right),\ \exp\left(\frac{2}{3}\pi i\right)z$

(7) $z+i,\ z+\exp\left(\frac{1}{6}\pi i\right),\ \exp\left(\frac{1}{6}\pi i\right)z,$

where p denotes a complex number with non-zero imaginary part and $|p| \geq 1$. One can find this classification in Ford [1].

Let E be an extended group with one limit point. Then it contains a Kleinian group G which has one of the above forms and G has index 2 in E. Hence E

can be written as

$$E=G\cup GU,$$

$$az=e^{i\theta}\bar{z}+b,\ 0\leq\theta<2\pi.$$

Let T be an element in G, then

$$T(z)=K^m z+d;$$

if $K^m=1$, then $T(z)=z+d$, and we call d the period of the parabolic element T. Let D be the set of periods of all parabolic elements contained in G. Let T_1 and T_2 be two elements in G and let U be an anti-analytic element in E, then

(8) U^2, UT_1UT_2, UT_1U, and $T_1UT_2U^{-1}$

are contained in G. From the above fact we conclude

(9) $e^{i\theta}\bar{D}\subset D$, $e^{i\theta}\ \bar{b}+b\in D$. $-\bar{K}^m b+b\in D$

$$\bar{K}^m=K^l,\ e^{i\theta}\bar{K}^m\in D \text{ and } \bar{K}^m K^n b+\bar{K}^n b\in D$$

where m and n are integers.

Let $E=G\cup GU$ and G be generated by $z+1$. Then by the condition $e^{i\theta}\ \bar{D}\subset D$, we conclude $\theta=0$ or $\theta=\pi$. Let $\theta=0$ and $B(z)=z+a$ then $BGB^{-1}=G$ and $BUB^{-1}=\bar{z}-\bar{a}+b+a$.

Considering BEB^{-1}, we know that there are two cases $b=0$ or $b=\frac{1}{2}$ and hence there are two non-conjugate groups;

$$(z+1,\ \bar{z}) \text{ and } \left(z+1,\ \bar{z}+\frac{1}{2}\right).$$

In case $\theta=\pi$ we have $-\bar{b}+b\in D$ by (9) and hence b must be a real number.

Conjugating by an element of type $z+a$ we obtain $b=0$. Hence the group E is generated by

$$(z+1,\ -z).$$

Now G is generated by (2), by calculation similar to the above we have two groups;

$$(z+1,\ z+p,\ -\bar{z}) \text{ and } \left(z+1,\ z+p,\ -\bar{z}+\frac{1}{2}\right).$$

Let G be generated by (4), (5), (6) or (7), G contains a doubly periodic subgroup, and if we know the parallelograms generated by the subgroup then by the condition $e^{i\theta}\bar{D}\subset D$, we can determine θ and b.

Let $p\neq0$ be a period of a parabolic element in G; if there is no period q of a parabolic element in G such that

$$nq=\pm p,$$

where $n\geq2$ is an integer, then we say p is a *minimal period*. If p is a minimal period then so is $-p$, hence we consider these two to be the same minimal periods, so that there are four minimal periods for a given D. Let us denote the four minimal periods by l, p, q and r.

Let G be generated by $z+1$ and suppose D satisfies (11),

(11) $1<|p|=|q|<|r|$, where p, q and r are not imaginary.

Then $e^{i\theta}\bar{D}\subset D$ implies $\theta=0$ or $\theta=\pi$. If $\theta=0$ then conjugating by $B(z)=z+a$, we have $b=0$ or $b=\frac{1}{2}$. Hence we have two groups

$(z+1,\ z+p,\ \bar{z})$ and $\left(z+1,\ z+p,\ \bar{z}+\frac{1}{2}\right)$.

If $\theta=\pi$, then by $e^{i\theta}\bar{b}+b\in D$, we have $-\bar{b}+b\in D$, and hence b must be a real. We have two cases $b=0$ or $b=\frac{1}{2}$, but by a conjugation with $B(z)=z+a$ we can take $b=0$ always. Hence E is generated by

$$(z+1,\ z+p,\ -\bar{z}).$$

Since the proof is similar in each case, we give only results. Consider the following classifications of minimal periods;

(10) $1<|p|<|q|<|r|$

(11) $1<|p|=|q|<|r|$

(12) $1=|p|<|q|\leq|r|$, $p=e^{i}\alpha$

(13) $1=|p|<|q|=|r|$, $p=e^{2/6}\ \pi^{i}$

(14) $1<|p|<|q|=|r|$, $p=i$

(15) $1<|p|<|b|=|r|$, $p=si$, $s\neq 0$ is real.

Let G be generated by (3) and let $E=G\cup GU$. We give a table of anti-analytic generators of E corresponding to the above table of minimal periods.

Periods, U

(10) do not exist

(11) $\bar{z}$, $\bar{z}+12$, $-\bar{z}$

(12) $e^{i\alpha}\bar{z}$, $e^{i\alpha+\pi}\bar{z}$

(13) $\bar{z}+\frac{1}{2}$, $e^{i\theta}\bar{z}\left(\text{where } \theta=\frac{k}{6}2\pi i,\ k=0,\ 1,\dots 5.\right)$

(14) $\bar{z}+\frac{1}{2}$, $e^{i\theta}\bar{z}\left(\text{where } \theta=\frac{k}{4}2\pi i,\ k=0,\ 1,\ 2, 3.\right)$

(15) $\bar{z}$, $\bar{z}+\frac{1}{2}-\bar{z}$.

Let G be generated by (4), then we have the following table of U.

Periods, U.

(10) do not exist

(11) $\bar{z},\ \bar{z}+\frac{1}{2}$

(12) $\bar{a},\ \bar{a}+\frac{1}{2}(1+e^{i\alpha})$

(13) $\bar{z},\ \bar{z}+\frac{1}{2},\ e^{i\theta}\bar{z},\ e^{i\alpha}\bar{z}+(1+e^{i6}),\ 0=\frac{\pi}{3}$ or $\frac{2}{3}\pi$

(14) $\bar{z},\ \bar{z}+\frac{1}{2},\ i\bar{z}+\frac{1}{2}(1+i)$

(15) $\bar{z},\ \bar{z}+\frac{1}{2},\ \bar{z}+\frac{1}{2}si.$

Let G be generated by (5), then U takes two terms: $\bar{z}$ and $i\bar{z}+\frac{1}{2}$, $i\bar{z}+\frac{1}{2}$ $(1+i)$.

Let G be generated by (6), then U takes the two terms $\bar{z}$ and $e^{1/3\pi i}\ \bar{z}$.

Let G be generatde by (7), then U is $\bar{z}$.

2. Groups with two limit points

Now we consider groups with two limit points; these can be classified into four kinds of groups, see Ford [1]. As before, we represent them by generators:

(16) Kz

(17) $Kz,\ K_1z$

(18) $Kz,\ K/z$

(19) $Kz,\ K/z,\ K_1z,\ K_1/z,$

where $|K|\neq 1$ and $K_1=\exp(2\pi i/k)$ and k is a positive integer.

Let $E=G\cup GW$ then W has two forms such that

$W=U=re^{i\theta}\bar{z}$ and $W=V=(re^{i\theta}/\bar{z})$

Now consider the following elements of E;

(20) $U^2=\rho^2 z$

(21) $(UT^m)(UT^n)=\rho^2\bar{K}^m K^n z$ where $T^n=K^n z$.

(22) $V^2=e^{2i\theta}z$

(23) $VT^nV^{-1}=\bar{K}^{-n}z.$

Let G be generated by (16). By (20) and (21) we have

(24) $\rho^2=K^l$ and $K^i\bar{K}^m K^n=K^s$

where $l,\ m,\ n$ and s are integers.

Set $K=|K|e^{i\alpha}$, by (24) we have $\alpha=0$. Let $S=Me^{-1/2i\theta}$ where $M>0$, then $SUS^{-1}=V\bar{z}$ and $SGS^{-1}=G$. Hence we can set $\theta=0$. Consiering T^nU we can take $V=1$ or $V^2=K$. Hence we have two groups.

$(Kz,\ \bar{z})$ and $(Kz,\ K^{1/2}\bar{z}), K>0.$

Let us consider V, let $S=V^{1/2}e^{i\alpha}\bar{z}$ then we have

$$SGS^{-1}=G \text{ and } SVS^{-1}=e^{i\theta}/\bar{z}.$$

By (23) we know that K is positive and real, and by (22) We conclude $\theta=0$ or π. Hence we have the following two groups;

$$(Kz,\ 1/\bar{z}) \text{ and } (Kz,\ -1/\bar{z}).$$

Repeating similar calculations we have the following results.

Let G be (17); then W takes the following forms;

$\bar{z},\ K^{1/2}\bar{z},\ 1/\bar{z}$ and $-1/\bar{z}$, where $K>0$.

$\bar{z},\ 1/\bar{z},\ -1/\bar{z}$ and $e^{\pi/k}/\bar{z}$, where $K=|K|\ e^{\pi i/k}$

Let G be (18), then W takes the forms;

$\bar{z},\ -\bar{z},\ K^{1/2}\bar{z},\ -K^{1/2}\bar{z}$, where $K>0$.

Let G be as in (19), then W takes the forms;

$Ve^{i\theta}\bar{z}$, where $V=1$ or $V=\sqrt{|K|}$

and $\theta=0$, $\theta=\pi$ or $\theta=\pi/k$ and $K=|K|\ e_{\pi i/k}$ or $K=$real.

For the classifications of extended groups with no limit point, one can refer to Ford [1], he gives there a simple classification.

Kyungpook University,
State University of New York
at Stony Brook

REFERENCES

[1] L. Ford, *Automorphic functions*, second edition, Chesea publishing company, New York, (1951).

[2] J. Lehner, *A short course in automorphic functions*,, Holt, Reinhart and Winston, New York, (1965).