WELL-CHAINED UNIFORMITIES

by Norman Levine

1. Introduction

DEFINITION 1.1. A uniform space $(X, \mathscr{U})$ is well-chained and $\mathscr{U}$ is a well-chained uniformity iff x and y in X and U in $\mathscr{U}$ implies that there exist $x_1, \cdots, x_n$ in X such that $x_1=x$, $x_n=y$ and $(x_i, x_{i+1})\in U$ for $1\leq i\leq n-1$. $x_1, \cdots, x_n$ is called a U-chain from x to y.

The following two theorems appear in several texts on topology (see [1], [2]):

THEOREM 1.2. *$(X, \mathscr{U})$ is a well-chained uniform space if $(X, \mathscr{T}(\mathscr{U}))$ is a connected topological space.*

THEOREM 1.3. *$(X, \mathscr{T}(\mathscr{U}))$ is a connected space if $(X, \mathscr{U})$ is compact and well-chained.*

In §4, we show that a completely regular space $(X, \mathscr{T})$ is connected iff $(X, \mathscr{U})$ is well-chained for every uniformity $\mathscr{U}$ which generates $\mathscr{T}$. Several corollaries are given.

In §2, we show that a space $(X, \mathscr{U})$ is well-chained iff $X\times X$ is the only equivalence relation in $\mathscr{U}$. Other characterizations of well-chained are obtained.

In §3, we show that a dense subspace Y of a uniform space X is well-chained iff the uniform space X is well-chained.

In §5, we show that the uniformly continuous image of a well-chained space is well-chained; we show that a product space is well-chained iff each factor space is well-chained.

In §6, we show that a uniform space $(X, \mathscr{U})$ is well-chained if its hyperspace is well-chained; we show that a certain subspace of the hyperspace is well-chained if the space $(X, \mathscr{U})$ is well-chained.

Finally, in §7, we show that a certain function space is well-chained.

2. Characterizations of well-chained uniformities.

LEMMA 2.1. *Let X be a set and $\triangle\subset V\subset X\times X$ where $\triangle$ is the diagonal and $V=V^{-1}$. Then $\bigcup\{V^n : n\geq 1\}$ is an equivalence relation, V^n being $V^{n-1}\circ V$.*

THEOREM 2.2. *A space $(X, \mathscr{U})$ is well-chained iff $X \times X$ is the only equivalence relation in $\mathscr{U}$.*

PROOF. A. Suppose that $(X, \mathscr{U})$ is well-chained and that $E \in \mathscr{U}$, E being an equivalence relation. We will show that $X \times X \subset E$; let $(x, y) \in X \times X$. There exist then an E-chain $x_1, \cdots\cdots, x_n$ from x to y. Then $(x, y) \in E^{n-1} = E$.

B. Let $X \times X$ be the only equivalence relation in $\mathscr{U}$ and suppose that $(x, y) \in X \times X$ and that $U \in \mathscr{U}$. Let $V \subset U$, $V \in \mathscr{U}$, $V = V^{-1}$ and $E = \bigcup\{V^n : n \geq 1\}$. By lemma 2.1, E is an equivalence relation and hence $E = X \times X$. Thus $(x, y) \in E$ and hence $(x, y) \in V^n$ for some n. Hence there exists a V-chain from x to y. This V-chain is also a U-chain.

COROLLARY 2.3. *A space $(X, \mathscr{U})$ is well-chained iff $X \times X = \bigcup\{U^n : n \geq 1\}$ for each $U \in \mathscr{U}$.*

PROOF. A. Let $(X, \mathscr{U})$ be well-chained and suppose $U \in \mathscr{U}$. Let $V \subset U$, $V = V^{-1}$, $V \in \mathscr{U}$ and $E = \bigcup\{V^n : n \geq 1\}$. By lemma 2.1, E is an equivalence relation in $\mathscr{U}$ and by theorem 2.2, $E = X \times X$. It follows then that $X \times X = \bigcup\{U^n : n \geq 1\}$.

B. Let $X \times X = \bigcup\{U^n : n \geq 1\}$ for each $U \in \mathscr{U}$. To show that $(X, \mathscr{U})$ is well-chained, by theorem 2.2, it suffices to show that $X \times X$ is the only equivalence relation in $\mathscr{U}$. Let $E \in \mathscr{U}$, E being an equivalence relation. Then $E = \bigcup\{E^n : n \geq 1\} = X \times X$.

COROLLARY 2.4. *$(X, \mathscr{U})$ is well-chained iff $\phi \neq A \neq X$, $U \in \mathscr{U}$ implies that $U[A] \cap \mathscr{C}A \neq \phi$.*

PROOF. A. Let $(X, \mathscr{U})$ be well-chained and suppose that $\phi \neq A \neq X$, $U \in \mathscr{U}$ and that $U[A] \cap \mathscr{C}A = \phi$. There exists a $V \in \mathscr{U}$, $V = V^{-1}$ *and* $V \circ V \subset U$. It is easy to see that $V[A] \cap V[\mathscr{C}A] = \phi$. Let $E = V[A] \times V[A] \cup V[\mathscr{C}A] \times V[\mathscr{C}A]$. Then E is an equivalence relation and $E \neq X \times X$ as the reader can easily check. By theorem 2.2, we arrive at a contradiction by showing that $E \in \mathscr{U}$; assert $V \subset E$. Let $(x, y) \in V$. We may assume that $x \in A$. Then $y \in V[A]$ and $(x, y) \in V[A] \times V[A] \subset E$.

B. Let $U[A] \cap \mathscr{C}A \neq \phi$ for all $U \in \mathscr{U}$ and all $\phi \neq A \neq X$. We show that $(X, \mathscr{U})$ is well-chained. Let $E \in \mathscr{U}$, E being an equivalence relation. By theorem 2.2, it suffices to show that $E = X \times X$. Suppose $E \neq X \times X$. Let $(x, y) \notin E$ and let $A = E[x]$. Then $\phi \neq A \neq X$ and $E \in \mathscr{U}$. But $E[A] \cap \mathscr{C}A = A \cap \mathscr{C}A = \phi$, a contradiction.

COROLLARY 2.5. *A space $(X, \mathscr{U})$ is not well-chained iff there exist disjoint, non empty open sets O_1 and O_2 such that $X = O_1 \cup O_2$ and $(O_1 \times O_1) \cup (O_2 \times O_2) \in \mathscr{U}$.*

PROOF. A. Suppose $X=O_1 \cup O_2$, $(O_1 \times O_1) \cup (O_2 \times O_2) \in \mathscr{U}$, O_i being disjoint, non-empty open sets. Then $(O_1 \times O_1) \cup (O_2 \times O_2)$ is an equivalence relation in $\mathscr{U}$ which is different from $X \times X$ and by theorem 2.2, $(X, \mathscr{U})$ is not well-chained.

B. Suppose $(X, \mathscr{U})$ is not well-chained. By corollary 2.4, there exists $\phi \neq A \neq X$ and a $U \in \mathscr{U}$ such that $U[A] \cap \mathscr{C}A = \phi$. It is easy to see that A is both open and closed. It suffices then to show that $A \times A \cup \mathscr{C}A \times \mathscr{C}A \supset U \cap U^{-1}$. Suppose $(x, y) \in U \cap U^{-1}$, but $(x, y) \notin (A \times A) \cup (\mathscr{C}A \times \mathscr{C}A)$. We may assume that $x \in \mathscr{C}A$ and $y \in \mathscr{C}A$. Then $y \in U[A] \cap \mathscr{C}A$, a contradiction.

3. Subspaces.

THEOREM 3.1. *Let $(Y, \mathscr{V})$ be a dense subspace of $(X, \mathscr{U})$. Then $(Y, \mathscr{V})$ is well-chained iff $(X, \mathscr{U})$ is well-chained.*

PROOF. A. Let $(Y, \mathscr{V})$ be well-chained and let a, b be in X and U in $\mathscr{U}$ with $U=U^{-1}$. Then $Y \cap U[a] \neq \phi \neq Y \cap U[b]$ since Y is dense in X. Take $c \in Y \cap U[a]$ and $d \in Y \cap U[b]$ and let $V=Y \times Y \cap U$. There exists a V-chain $y_1, \cdots, y_n$ from c to d and hence $a, y_1, \cdots, y_n, b$ is a U-chain from a to b.

B. Let $(X, \mathscr{U})$ be well-chained, a, b in Y and $V \in \mathscr{V}$. Then $V=Y \times Y \cap U$ for some $U \in \mathscr{U}$; let $W \in \mathscr{U}$ and $W \circ W \circ W \subset U$, $W=W^{-1}$. Now there exists a W-chain $x_1, \cdots, x_n$ from a to b. Choose $y_i \in W[x_i] \cap Y$ for each i. Then $a, y_1, \cdots, y_n, b$ is a V-chain from a to b.

4. Connectedness.

THEOREM 4.1. *Let $(X, \mathscr{T})$ be a completely regular topological space. Then $(X, \mathscr{T})$ is connected iff $(X, \mathscr{U})$ is well-chained for every $\mathscr{U}$ such that $\mathscr{T}=\mathscr{T}(\mathscr{U})$.*

PROOF. A. If $(X, \mathscr{T})$ is connected, then $(X, \mathscr{U})$ is well-chained for all $\mathscr{U}$ for which $\mathscr{T}=\mathscr{T}(\mathscr{U})$. This is the statement of theorem 1.2.

B. Suppose $(X, \mathscr{T})$ is disconnected; then there exist non-empty disjoint open sets O_1 and O_2 such that $X=O_1 \cup O_2$. Let $E=(O_1 \times O_1) \cup (O_2 \times O_2)$ and let $\mathscr{V}$ be the uniformity for X consisting of all supersets of E. Let $\mathscr{U}$ be any uniformity for X such that $\mathscr{T}=\mathscr{T}(\mathscr{U})$. Then $\mathscr{U} \vee \mathscr{V}$ generates $\mathscr{T}$; but by corollary 2.5, $\mathscr{U} \vee \mathscr{V}$ is not well-chained since $E \in \mathscr{U} \vee \mathscr{V}$.

COROLLARY 4.2. *Let $(X, \mathscr{U})$ be a fine uniform space. Then $(X, \mathscr{T}(\mathscr{U}))$ is connected iff $(X, \mathscr{U})$ is well-chained.*

PROOF. This follows from the fact that a subuniformity of a well-chained uniformity is well-chained.

COROLLARY 4.3. *Let $(X, \mathscr{U})$ be a uniformity with the property that $U \in \mathscr{U}$ iff U is a neighborhood of the diagonal. Then $(X, \mathscr{T}(\mathscr{U}))$ is connected iff $(X, \mathscr{U})$ is well-chained.*

PROOF. $\mathscr{U}$ is a fine uniformity.

COROLLARY 4.4. *Let $(X, \mathscr{T})$ be paracompact and regular. If $\mathscr{U}$ consists of all neighborhoods of the diagonal, then* (i) *$\mathscr{U}$ is a uniformity for $\mathscr{T}$ and* (ii) *$(X, \mathscr{T})$ is connected iff $(X, \mathscr{U})$ is well-chained.*

PROOF. (i) is well known and (ii) follows from corollary 4.3.

5. Transfers of well-chained uniformities and product spaces.

THEOREM 5.1. *Let $f: (X, \mathscr{U}) \to (Y, \mathscr{V})$ be a uniformly continuous surjection. If $(X, \mathscr{U})$ is well-chained, then $(Y, \mathscr{V})$ is well-chained.*

PROOF. Let E be an equivalence relation in Y and suppose that $E \in \mathscr{V}$. By theorem 2.2, it suffices to show that $E = Y \times Y$. Now $(f \times f)^{-1} E \in \mathscr{U}$ and $(f \times f)^{-1} E$ is an equivalence relation in X and since $(X, \mathscr{U})$ is well-chained, it follows that $(f \times f)^{-1} E = X \times X$. Then $E = Y \times Y$.

THEOREM 5.2. *Let $f: (X, \mathscr{U}) \to (Y, \mathscr{V})$ be a uniformly continuous surjection and suppose that $\mathscr{U}$ is the weak uniformity, that is, $\mathscr{U}$ has $\{(f \times f)^{-1} V : V \in \mathscr{V}\}$ as base. If $(Y, \mathscr{V})$ is well-chained, then $(X, \mathscr{U})$ is well-chained.*

PROOF. Let a, b be in X and suppose $V \in \mathscr{V}$. We show that there is an $(f \times f)^{-1} V$-chain from a to b. There exists a V-chain $y_1, \cdots\cdots, y_n$ from $f(a)$ to $f(b)$. Let $x_1 = a$, $x_n = b$ and $f(x_i) = y_i$ for $2 \leq i \leq n-1$. Then $(x_i, x_{i+1}) \in (f \times f)^{-1} V$ for $1 \leq i \leq n-1$.

LEMMA 5.3. *Let $(X, \mathscr{U})$ be a uniform space and $x_1, \cdots\cdots, x_n$ be a U-chain from x to y. Then $x_1, \cdots\cdots, x_n, x_{n+1}, \cdots\cdots, x_m$ is a U-chain from x to y where $x_i = y$ for $n+1 \leq i \leq m$. In other words, a U-chain from x to y can be entended to any finite length.*

THEOREM 5.4. *Suppose $(X, \mathscr{U}) = \times\{(X_\alpha, \mathscr{U}_\alpha) : \alpha \in \triangle\}$. Then $(X, \mathscr{U})$ is well-chained iff $(X_\alpha, \mathscr{U}_\alpha)$ is well-chained for each $\alpha \in \triangle$.*

PROOF: A. If $(X, \mathscr{U})$ is well-chained, then $(X_\alpha, \mathscr{U}_\alpha)$ is well-chained for each $\alpha \in \triangle$ by theorem 5.1.

B. Suppose that $(X_\alpha, \mathscr{U}_\alpha)$ is well-chained for each $\alpha\in\Delta$, x and y in X and $U=(P_{\alpha_1}\times P_{\alpha_1})^{-1}U_{\alpha_1}\cap\cdots\cap(P_{\alpha_k}\times P_{\alpha_k})^{-1}U_{\alpha_k}$. Then $x(\alpha_i)$, $y(\alpha_i)$ are in X_{α_i} and $U_{\alpha_i}\in\mathscr{U}_{\alpha_i}$. There exists then a U_{α_i}-chain $x_{\alpha_i}{}^1$, $x_{\alpha_i}{}^2$, ..., $x_{\alpha_i}{}^{n_i}$ from $x(\alpha_i)$ to $y(\alpha_i)$ for $1\leq i\leq k$. By lemma 5.3, we may assume that $n_1=n_2=\cdots=n_k=n$. Let $x_1=x$, $x_n=y$. For $2\leq i\leq n-1$, let $x_i:\Delta\to\bigcup\{x_\alpha : \alpha\in\Delta\}$ as follows: $x_i(\alpha_j)=x^i_{\alpha_j}$ for $1\leq j\leq k$ and $x_i(\alpha)\in X_\alpha$ arbitrary when $\alpha\neq\alpha_1, \cdots, \alpha_k$. It is easy to see that $x_1, \cdots, x_n$ is a U-chain from x to y.

6. Hyperspaces

DEFINITION 6.1. *For a set* X, *define* $H(X)=\{A : A\subset X,\ A\neq\phi\}$. *For* $\Delta\subset U\subset X\times X$, *let* $H(U)=\{(A, B) : A\in H(X),\ B\in H(X),\ A\subset U[B],\ B\subset U[A]\}$.

DEFINITION 6.2 *Let* $(X, \mathscr{U})$ be a uniform space. Then $H(\mathscr{U})$ is the uniformity for $H(X)$ generated by $\{H(U) : U\in\mathscr{U}\}$ as base. $(H(X), H(\mathscr{U}))$ is the ***hyperspace*** determined by $(X, \mathscr{U})$.

THEOREM 6.3. $(X, \mathscr{U})$ *is well-chained if* $(H(X), H(\mathscr{U}))$ *is well-chained.*

PROOF. Let x, y be in X and $U=U^{-1}$ be in $\mathscr{U}$. Then $\{x\}$, $\{y\}$ are in $H(X)$ and $H(U)\in H(\mathscr{U})$. There exist then A_1, ..., A_n in $H(X)$, a $H(U)$-chain from $\{x\}$ to $\{y\}$. Now $\{x\}=A_1\subset U[A_2]$; let $x_1=x$ and $x_2\in A_2$ such that $(x_1, x_2)\in U$. But $A_2\subset U[A_3]$; choose x_3 in A_3 such that $(x_2, x_3)\in U$. Continuing, we have $x_i\in A_i$ and $(x_i, x_{i+1})\in U$ for $1\leq i\leq n-1$. It follows that $x_1, \cdots, x_n$ is a U-chain from x to y.

The converse of theorem 6.3 is false as seen by

EXAMPLE 6.4. Let $(X, \mathscr{U})$ be the space of reals with the usual uniformity $\mathscr{U}$. Then $(X, \mathscr{U})$ is well-chained, but $(H(X), H(\mathscr{U}))$ is not well-chained since there is no $H(V_1)$-chain from $\{x\}$ to X where $V_1=\{(x, y) : |x-y|<1\}$.

DEFINITION 6.5. Let $A\in H(X)$, $(X, \mathscr{U})$ being a uniform space. Then A is *totally bounded* iff for each $U\in\mathscr{U}$, there exist a_i in A such that $A\subset U[a_1, \cdots, a_n]$. We define $T(X)=\{A : A\in H(X)$ and A is totally bounded$\}$.

THEOREM 6.6. *If* $(X, \mathscr{U})$ *is well-chained, then* $(T(X), T(X)\times T(X)\cap H(\mathscr{U}))$ *is well-chained.*

We first prove some lemmas.

LEMMA 6.7. *Let* $(X, \mathscr{U})$ *be well-chained,* $A\in H(X)$, $U=U^{-1}\in\mathscr{U}$ *and* $x\in X$.

Then there exists an $H(U)$-chain from A to $A\cup\{x\}$.

PROOF. Let $a\in A$. Then there exists a U-chain $x_1, \cdots\cdots, x_n$ from a to x. Then $(A, A\cup\{x_2\})\in H(U)$, $(A\cup\{x_2\}, A\cup\{x_3\})\in H(U)$, $\cdots\cdots$, $(A\cup\{x_{n-1}\}, A\cup\{x\})\in H(U)$. Note that $A\cup\{x\}$ is totally bounded when A is totally bounded.

LEMMA 6.8. *Suppose that $(X, \mathscr{U})$ is well-chained and that x and y are in X. Let $U=U^{-1}\in\mathscr{U}$. Then $\{x\}$ and $\{y\}$ are $H(U)$-chained.*

PROOF. Let $x_1, \cdots\cdots, x_n$ be a U-chain from x to y. Then $\{x_1\}, \cdots\cdots, \{x_n\}$ is an $H(U)$-chain from $\{x\}$ to $\{y\}$. Note that $\{x_i\}\in T(X)$.

LEMMA 6.9. *Let $(X, \mathscr{U})$ be well-chained, $A=\{a_1, \cdots\cdots, a_n\}$. Then A and $\{a_1\}$ are $H(U)$-chained where $U=U^{-1}\in\mathscr{U}$.*

PROOF. By lemma 6.7, $\{a_1, \cdots\cdots, a_i\}$ and $\{a_1, \cdots\cdots, a_{i+1}\}$ are $H(U)$-chained.

Proof of theorem 6.6., Let A and B be in $T(X)$ and $U=U^{-1}\in\mathscr{U}$. By definition 6.5, there exist points a_i and b_i such that $A\subset U[a_1, \cdots, a_n]$ and $B\subset U[b_1, \cdots, b_m]$ with a_i in A and b_i in B. It follows then that $(A, \{a_1, \cdots\cdots, a_n\})$ and $(B, \{b_1, \cdots\cdots, b_m\})$ are in $H(U)$. By lemma 6.9, $\{a_1, \cdots\cdots, a_n\}$ and $\{a_1\}$ are $H(U)$-chained and $\{b_1, \cdots\cdots, b_m\}$ and $\{b_1\}$ are $H(U)$-chained. By lemma 6.8, $\{a_1\}$ and $\{b_1\}$ are $H(U)$-chained. It is clear that all chains may be taken in $T(X)$. Thus $(T(X), T(X)\times T(X)\cap H(\mathscr{U}))$ is well-chained.

COROLLARY 6.10. Let $(X, \mathscr{U})$ be a totally bounded uniform space. Then $(X, \mathscr{U})$ is well-chained iff $(H(X), H(X)\times H(X)\cap H(\mathscr{U}))$ is well-chained.

PROOF. This follows from theorem 6.3 and the fact that $H(X)=T(X)$ when $(X, \mathscr{U})$ is totally bounded.

7. Function spaces.

Let $X\neq\phi$, $(Y, \mathscr{V})$ a uniform space and $\phi\neq\mathscr{F}\subset Y^X$. For $V\in\mathscr{V}$, let $W(V)=\{(f, g) : (f(x), g(x))\in V$ for all $x\in X$ and f and g in $\mathscr{F}\}$. Let $\mathscr{U}_{uc}(\mathscr{F})$ be the uniformity for $\mathscr{F}$ with $\{W(V) : V\in\mathscr{V}\}$ as base.

THEOREM 7.1. *Let $(Y, \mathscr{V})$ be a subspace of the reals with the usual uniformity and suppose that $X\neq\phi$. Suppose that $\mathscr{F}$ is a non-empty subset of the bounded functions in Y^X with the property that f, g in $\mathscr{F}$ implies that $\frac{1}{2}(f+g)$ is in $\mathscr{F}$. Then $(\mathscr{F}, \mathscr{U}_{uc}(\mathscr{F}))$ is well-chained.*

PROOF. Let f and g be in $\mathscr{F}$ and suppose $V\in\mathscr{V}$. Then there exists an $\varepsilon>0$

such that $V_\varepsilon \subset V$, $V_\varepsilon = \{(y_1, y_2) : y_i \in Y \text{ and } |y_1 - y_2| < \varepsilon\}$.

Let $M = \sup\{|f(x) - g(x)| : x \in X\}$ and choose n so that $M/2^n < \varepsilon$.

Let $f_1 = f$, $f_2 = f_1 + (g-f)/2^n$, and $f_{i+1} = f_i + (g-f)/2^n$.

Then $f_{2^n+1} = g$ and $(f_i, f_{i+1}) \in W(V_\varepsilon)$ for $1 \leq i \leq 2^n$. It is clear that $f_i \in \mathscr{F}$ for each i.

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REFERENCES

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