ON LOCALCOMPACTIFICATIONS OF TYCHONOFF SPACES

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Being locally compact is not a reflective property in the sense of Herrlich and Van der Slot [2] since it is not productive. And so we cannot talk of largest locally compact extensions of spaces in the usual sense. However the study of 'localcompactifications' is important since if we assume further that a localcompactification Y of X is Hausdorff, then Y becomes Tychonoff and so there exist sufficiently many real-valued continuous functions on it. In this paper, we prove that the set $L(X)$ of all localcompactifications of a space X under a suitable order forms a complete upper semi-lattice and the semi-lattice $K(X)$ of all Hausdorff compactifications of X is a sub-semi-lattice of $L(X)$. $L(X)$ is never a lattice but when X is locally compact, there exist minimal elements. Further we show that under a suitable equivalence, the localcompactifications of X are nothing but the open subspaces of the Stone-Čech compactification βX of X each of them containing X.

It follows from a theorem of Magill, K.D., Jr. [3] on lattices of compactifications that if X and Y are locally compact, then $L(X)$ and $L(Y)$ are semi-lattice isomorphic if and only if $\beta X-X$ and $\beta Y-Y$ are homomorphic.

CONVENTION. All spaces considered are completely regular.

DEFINITION 1. *Let X be a space. A space Y is said to be a localcompactification of X if Y is locally compact and X is dense in Y*

RESULT 2. *A space X is locally compact if and only if it is the intersection of a closed subspace and an open subspace of I^F where $I=[0,1]$ with usual topology, for some indexing set F.*

PROOF. Any closed (or open) subspace of a locally compact space is locally compact. So sufficiency follows. Conversely, let X be a locally compact space. Let αX be any compactification of X. Then αX is homeomorphic to a closed subspace Y of I^F for some F. X is open in αX. So X is the intersection of some open subspace of I^F with Y. Thus necessity.

RESULT 3. *A space Y is a localcompactification of X if and only if it is an open subspace (containing X) of some compactification of X.*

PROOF. If eX is a localcompactification of X, consider βeX, the Stone-Čech compactification of eX. It is clearly a compactification of X and eX being locally compact and dense in βeX, is open in βeX. Converse is trivial.

NOTE. A subset A of a space X is said to be C^*-embedded in X if every bounded continuous real-valued function on A extends continuously to X ([1]).

RESULT 4. *Let $\alpha X \in K(X)$. All open C^*-embedded subsets of αX each containing X, form a complete upper semi-lattice under inclusion, with αX as the 1-element. If X is C^*-embedded in αX i.e., if α^X = the Stone-Čech compactification βX of X, then these form a lattice if and only if X is locally compact, in which case the lattice is complete. More generally, if there is a smallest C^*-embedded subset eX of αX such that $X \subset eX$, then the open C^*-embedded subsets of αX each containing X form a complete upper semi-lattice which is a lattice if and only if eX is locally compact in which case the lattice is complete.*

The proof is easy and is omitted.

NOTE. The collection of all localcompactifications of a Tychonoff space X modulo homeomorphisms with identity on X, is a set and we denote it by $L(X)$.

LEMMA 5. *$L(X)$ is setwise equivalent to $K(X) \cup \{eX \mid eX$ is an open C^*-embedded subset of some $\alpha X \in K(X)\}$.*

PROOF. Each element eX in $L(X)$ is either a compactification or an open C^*-embedded subset of βeX; Conversely any open C^*-embedded subset of any αX in $K(X)$ containing X is an element of $L(X)$. The association is 1-1 since if eX is a C^*-embedded open subset containing X of some $\alpha X \in K(X)$ and so $\alpha' X \in K(X)$ then αX and $\alpha' X$ are equivalent as compactification of eX and so as compactification of X also. Thus the lemma.

DEFINITION 6. Let eX, $e'X \in L(X)$. We define $eX \geq e'X$ as follows:

(i) If eX and $e'X$ are in $K(X)$, then $eX \geq e'X$ if and only if they are so in $K(X)$.

(ii) If $eX \in K(X)$ and $e'X \notin K(X)$, then there exists a unique $\alpha X \in K(X)$ such that $e'X$ is an open C^*-embedded subset of αX. If $\alpha X \leq eX$, then $e'X \leq eX$; otherwise they are not comparable.

(iii) If eX, $e'X \notin K(X)$, then there exists αX, $\alpha' X \in K(X)$ uniquely specified such that eX, $e'X$ are respectively open C^*-embedded in αX, $\alpha' X$. If αX and $\alpha' X$ are comparable in $K(X)$ say, $\alpha X \leq \alpha' X$, then let f: $\alpha' X \rightarrow \alpha X$ be the unique continuous map with identity on X. We say $eX \leq$ (or $>$) $e'X$ according as $f^{-1}(\alpha X)$ $\subset$(or $\supsetneqq$) $e'X$. If αX, $\alpha' X$ are not comparable, then $eX, e'X$ also not comparable.

Now with respect this order in $L(X)$, we have:

THEOREM 7. *$L(X)$ is a complete upper semi-lattice and $K(X)$ is a sub semi-lattice of $L(X)$. $L(X)$ is never a lattice. However there exist minimal elements if X is locally compact.*

The proof is easy and omitted.

THEOREM 8. *Under suitable equivalence, the localcompactifications of a Tychonoff space X are nothing but the open subspaces of βX which contain X, where βX is the Stone-Čech compactification of X.*

PROOF. Let e_1X, $e_2X \in L(X)$. Let $f_{1\beta}$, $f_{2\beta}$ be the canonical continuous maps from βX onto βe_1X and βe_2X each being identity on X. If $f_{1\beta}^{-1}(e_1X) = f_{2\beta}^{-1}(e_2X)$ as subsets of βX, then we define $e_1X \sim e_2X$. Clearly $\sim$ is an equivalence relation. Under this equivalence relation, the localcompactifications of X are nothing but the open subspaces of βX which contain X.

THEOREM. 9. *Let X and Y be locally compact. Then $L(X)$ and $L(Y)$ are isomorphic as semi-lattices if and only if $\beta X - X$ and $\beta Y - Y$ are homeomorphic.*

This follows from the corresponding theorem on lattices of compactifications due to Magill K. D., Jr. [3] and by result 3 above.

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REFERENCES

[1] Gillman, L. and Jerison, M., *Rings of continuous Functions*, (Van Nostrand), (1960).

[2] Herrlich, H. and Van der Slot, J., *Properties which are closely related to compactness* Nederl. Akad. Wetensch. Proc. Ser. A. 70 (1967) 524-529.

[3] Magill, K. D., Jr., *The lattice of compactifications of a locally compact space*, Proc. Lond. Math. Soc. 18 (1968) 231-244.