SEMI-DEVELOPABLE SPACES AND SEMI-STRATIFIABLE SPACES

By Moo Ha Woo

A developable space is a well known topological space. Recently Charles C. Alexander [1] introduced a semi-developable space. Ceder [6] introduced M_3-spaces and Borges [5] renamed them "stratifiable", while Geoffrey D. Creede [7] studied semi-stratifiable spaces.

In this paper, we shall give a relation between semi-developable spaces and semi-stratifiable spaces without an aid of any separation axioms and some properties of semi-developable spaces.

The notation and terminology used in this paper follow those of J. L. Kelley [9] mainly. N is the set of positive intgers.

DEFINITION 1. [1] A *development* for a space X is a sequence $\{g_n | n \in N\}$ of open covers of X such that $\{St(x, g_n) | n \in N\}$ is a local base at x, for each $x \in X$.

DEFINITION 2. [1] A *semi-development* for a space X is a sequence $\{g_n | n \in N\}$ of (not necessarily open) covers of X such that $\{St(x, g_n) | n \in N\}$ is a neighborhood base at x, for each $x \in X$.

A space is (semi-) developable if and only if there exists a (semi-) development for the space.

DEFINITION 3. [7] A topological space X is *semi-stratifiable* if, to each open set $U \subset X$, one can assign a sequence $\{U_n | n \in N\}$ of closed subsets of X such that

(a) $\bigcup_{n=1}^{\infty} U_n = U$

(b) $U_n \subset V_n$ whenever $U \subset V$.

A correspondence $U \longrightarrow \{U_n | n \in N\}$ is a *semi-stratification* for the space X whenever it satisfies the conditions of Definition 3.

Alexander [1] showed that a space is semi-metrizable if and only if it is a semi-developable T_0-space. Creede [7] showed that a T_1-space is a semi-metric space if and only if it is a first countable semi-stratifiable space.

THEOREM 1. *Every semi-developable space is semi-stratifiable.*

PROOF. Let X be a semi-developable space and $\Delta=\{g_n|n\in N\}$ be a semi-development for X. For each n and each open set $U\subset X$, we take $U_n=\overline{[St(U',g_n)]'}$. Then the correspondence $U\longrightarrow\{U_n|n\in N\}$ is a semi-stratification for X.

For, (a) $U=\bigcup_{n=1}^{\infty}U_n$: Let $y\in\bigcup_{n=1}^{\infty}U_n$, there is an integer m such that $y\in U_m$(i. e., $y\in\overline{[\mathrm{St}(U', g_m)]'}$). Suppose that $y\notin U$, then we have $\mathrm{St}(y, g_m)\subset \mathrm{St}(U', g_m)$. Therefore, we obtain $\mathrm{St}(y, g_m)\cap[\mathrm{St}(U', g_m)]'=\phi$. Thus, $y\notin\overline{[\mathrm{St}(U', g_m)]'}$. This is contradict to $y\in\overline{[\mathrm{St}(U', g_m)]'}$.

For each $y\in U$, there is an integer m such that $\mathrm{St}(y, g_m)\subset U$. Therefore we have $\mathrm{St}(y, g_m)\cap U'=\phi$. For such m, we obtain $y\notin\mathrm{St}(U', g_m)$. Thus, we have $y\in\overline{[\mathrm{St}(U', g_m)]'}$. $(i,e.,y\in U_m)$.

(b) If U,V be open sets in X such that $U\subset V$, then we have $St\ (U', g_n)\supset \mathrm{St}(V', g_n)$ for each n. Hence we obtain $\overline{[\mathrm{St}(U', g_n)]'}\subset\overline{[\mathrm{St}(V', g_n)]'}$.

REMARK. The converse of Theorem 1 is not true since a semi-developable space is first countable but a semi-stratifiable space is not.

A topological space is *F_σ-screenable* if every open cover has a σ-discrete closed refinement which covers the space [7].

In [2], Bing showed that a developable space is F_σ-screenable. With the aid of Theorem 2.6 of [7] and Theorem 1, we have the following.

COROLLARY 2. *A semi-developable space is F_σ-screenable.*

In [1], Alexander showed that every Lindelöf semi-developable space is separable. Applying Theorem 1, we have the following.

PROPOSITION 3. *In a semi-developable space X, we have the followings*:

(1) *If X is normal, then X is perfectly normal.*

(2) *If X is Lindelöf, then X is hereditarily separable.*

(3) *If X is paracompact, then X is hereditarily paracompact.*

PROOF. (1) is trivial.

(2) A semi-developable space has hereditary property. Since a Lindelöf (paracompact) space is hereditarily Lindelöf (paracompact) if and only if each open subspace is Lindelöf (paracompact). (see [4], [8]).

Applying Theorem 1, X is semi-stratifiable. Let U be an open set in X, then $U=\bigcup_{n=1}^{\infty}U_n$ where U_n is closed in X. Hence U is Lindelof. Consequently, X is hereditarily separable.

(3) If an open subset U in X is a F_σ-set, then U is paracompact (see [8]). Hence it is trivial by Theorem 1.

Using a proof analogous to one given by Burke for Theorem 3.6 of [5], the following Theorem may be proved.

THEOREM 4. *A semi-stratifiable space X with a locally semi-development is semi-developable.*

PROOF. For each $x \in X$, there is an open neighborhood U_x with a semi-development $\Delta_x = \{g_n(x) \mid n \in N\}$. Since X is F_σ-screenable, there is a σ-discrete closed refinement $\mathscr{L} = \bigcup_{n=1}^{\infty} \mathscr{L}_n$ of $\{U_x \mid x \in X\}$.

For each $B \in \mathscr{L}_n$, there exists a fixed element $x(B) \in X$ such that $B \subset U_{x(B)}$.
Let $U_n(B) = X - \bigcup \{B^* \mid B^* \in \mathscr{L}_n,\ B^* \neq B\}$,

$\mathscr{U}_{n,m}(B) = \{U_n(B) \cap G \mid G \in g_m[x(B)]\}$ and

$\mathscr{U}_{n,m} = \{U \mid U \in \mathscr{U}_{n,m}(B),\ B \in \mathscr{L}_n\} \cup \{Q_n\}$ where $Q_n = X - \bigcup \{B \mid B \in \mathscr{L}_n\}$.

Then $\Delta = \{\mathscr{U}_{n,m} \mid n,\ m \in N\}$ is a semi-development for X.

For, it is trivial that $\mathscr{U}_{n,m}$ is a cover of X for each n, $m \in N$. Let $z \in X$, there is an integer $n \in N$ with some $B \in \mathscr{L}_n$ such that $z \in B$. If O be an open neighborhood of z, there is an integer m such that $z \in \text{Int St}\ [z,\ g_m(x(B))] \subset \text{St}[s,\ g_m(x(B))] \subset O \cap U_{x(B)}$.

Since $\text{Int St}[z,\ \mathscr{U}_{n,m}] = \text{Int St}\ [z,\ \mathscr{U}_{n,m}(B)]$

$$= \text{Int}\ [U_n(B) \cap \text{St}[z, g_m(x(B))]]$$
$$= U_n(B) \cap \text{Int St}[z,\ g_m(x(B))] \ni z,$$

hence we have $(n,\ m)$ such that $z \in \text{Int St}[z,\ \mathscr{U}_{n,m}] \subset \text{St}[z,\ \mathscr{U}_{n,m}] = \text{St}[z,\ \mathscr{U}_{n,m}(B)] \subset \text{St}[z,\ g_m(x(B))] \subset 0$.

For each $k,\ l \in N$, we have $z \in \text{Int St}\ (z,\ \mathscr{U}_{k,l})$. For, If there is B such that $z \in B \in \mathscr{L}_k$, we obtain $z \in \text{Int St}(z,\ \mathscr{U}_{k,l})$ by the above way. If there is not B such that $z \in B \in \mathscr{L}_k$, then $z \in Q_k$. Since Q_k is an open set, we have $z \in \text{Int St}\ (z,\ \mathscr{U}_{k,l})$. Hence $\{\text{St}(z,\ \mathscr{U}_{k,l}) \mid k,\ l \in N\}$ is a neighborhood base at z, for each $z \in X$.

Soong Jun College

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