THE STRUCTURE OF A CLASS OF REGULAR SEMIGROUPS, II

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We describe the structure of regular semigroups whose idempotents form a semigroup mod partial chains of left zero semigroups, partial chains of right zero semigroups, and inverse semigroups. Refer to [6] for a related structure theorem and a more complete bibliography.

If D is a semigroup, $E(D)$ will denote the set of idempotents of D. If $a \in D$, $\mathcal{I}(a)$ will denote the collection of inverses of a. Unless otherwise specified, we follow the definitions and notation of [1]. In particular, $\mathcal{R}$ and $\mathcal{L}$ will denote Green's relations.

Let W be a partial groupoid which is a union of a collection of pairwise disjoint subsemigroups $(T_{\delta} : \delta \in \Lambda)$ where Λ is a semilattice. If $x \in T_{\gamma}$, $y \in T_{\delta}$, and $\delta \leq \gamma$ (in Λ) imply xy is defined (in W) and $xy \in T_{\delta}$, and, if $\xi \leq \delta$ and $z \in T_{\xi}$ imply $(xy)z = x(yz)$. W is termed a (lower) partial chain of the semigroups $(T_{\delta} : \delta \in \Lambda)$. If $x \in T_{\gamma}$, $y \in T_{\delta}$, and $\gamma \leq \delta$ imply xy is defined (in W) and $xy \in T_{\gamma}$, and $\xi \geq \delta$ and $z \in T_{\xi}$ imply $(xy)z = x(yz)$, W is termed an (upper) partial chain of the semigroups $(T_{\delta} : \delta \in \Lambda)$. Partial chains of left groups were employed in [7].

Let M and N be sets. A mapping θ of a subset C of M into a subset F of N will be termed a partial mapping of M into N. We let $C = D(\theta)$ and $F = R(\theta)$. The set of all partial mappings of M into N is denoted by $P(M, N)$. We are now in a position to state our theorem.

Let X be an inverse semigroup with semilattice of idempotents Y. Let I be a (lower) partial chain of left zero semigroups $(I_y : y \in Y)$, and let J be an (upper) partial chain of right zero semigroups $(J_y : y \in Y)$. Let $(r, s) \to \alpha_{(r,s)}$ be a mapping of X^2 into $P(I \times J, I)$ and $(r, s) \to \beta_{(r,s)}$ be a mapping of X^2 into $P(I \times J, J)$ subject to the conditions

I. $D(\alpha_{(r,s)}) = D(\beta_{(r,s)}) = J_{r^{-1}r} \times I_{ss^{-1}}$; $R(\alpha_{(r,s)}) = I_{(rs)(rs)^{-1}}$; $R(\beta_{(r,s)}) = J_{(rs)^{-1}rs}$

II. If $j \in J_{s^{-1}s}$, $p \in I_{tt^{-1}}$, $q \in J_{t^{-1}t}$, and $m \in I_{gg^{-1}}$, $(j, p)\alpha_{(s,t)}((j, p)\beta_{(s,t)} \cdot q, m)\alpha_{(st,g)}$ $= (j, p((q, m)\alpha_{(t,g)}))\alpha_{(s,tg)}$ and

$$(j, p((q, m)\alpha_{(t,g)}))\beta_{(s,tg)}(q, m)\beta_{(t,g)} = ((j, p)\beta_{(s,t)}\, q, m)\beta_{(st,g)}.$$

Let (X, I, J, α, β) denote $\{(i, s, j) : s \in X, i \in I_{ss^{-1}}, j \in J_{s^{-1}s}\}$ under the multiplication

$$(i, s, j)(p, t, q) = (i((j, p)\alpha_{(s,t)}), st, (j, p)\beta_{(s,t)}q).$$

THEOREM. *S is a regular semigroup whose idempotents form a semigroup if and only if $S \cong (X, I, J, \alpha, \beta)$ for some collection X, I, J, α, β.*

PROOF. Let S be a regular semigroup such that $E(S)$ is a semigroup. By [2, theorem 3], $\lambda = \{(a, b) \in S : \mathcal{I}(a) = \mathcal{I}(b)\}$ is the smallest inverse semigroup congruence on S. Let $X = S/\lambda$, $Y = E(X)$, and $\lambda_s = s\lambda^{-1}$. Hence, $\{\lambda_s : s \in X\}$ is the collection of λ-classes of S and $\lambda_s\lambda_t \subset \lambda_{st}$. By [4 : 1, p. 129, Ex. 1], $E(S)$ is a semilattice Ω of rectangular bands $(E_\delta : \delta \in \Omega)$. Utilizing [3, lemma 2.2, 4], $\{\lambda : s \in Y\}$ is the collection of λ-classes of S containing idempotents. If $e, f \in E_\delta$ $(\delta \in \Omega)$, $\mathcal{I}(e) = \mathcal{I}(f)(\mathcal{I}(e) \cap \mathcal{I}(f)) \neq \phi$ implies $\mathcal{I}(e) = \mathcal{I}(f)$ by [2, theorem 2] and, hence, $E_\delta \subset \lambda_s$ for some $s \in Y$. If $h \in \lambda_s$, $\mathcal{I}(h) = \mathcal{I}(e)$, $h \in E(S)$ $(h \in \mathcal{I}(e))$ implies $h \in E(S)$ by [5, lemma 1.3] and, hence, $h \in E_\delta$. Thus, $E(S)$ is the semilattice Y of rectangular bands. $(\lambda_s : s \in Y)$. If $s \in Y$, select and fix an $\mathcal{L}$-class I_s of λ_s and select and fix an $\mathcal{R}$-class J_s of λ_s. For $s \in X$, let u_s denote a representative element of λ_s. If $e \in I_s$, $f \in I_t$, and $t \leq s$, $(ef, f) \in \mathcal{L}(\lambda_t)$ and, hence, $ef \in I_t$. Let $I = \cup(I_s : s \in Y)$ and, if $a, b \in I$, define $a \circ b = ab$ (product in S) if $ab \in I$ while $a \circ b$ is undefined if $ab \notin I$. Hence, the partial groupoid $(I, \circ)$ is a (lower) partial chain of left zero semigroups $(I_s : s \in Y)$ (since no confusion will arise, we replace "$\circ$" by juxtaposition). Similarly, $J = \cup(J_s : s \in Y)$ is an (upper) partial chain of right zero semigroups $(J_s : s \in Y)$. Noting the proof of [6, lemma 5], it is easily seen that every element of S may be uniquely expressed in the form $x = iu_sj$ where $i \in I_{ss^{-1}}$ and $j \in J_{s^{-1}s}$. Thus, if $j \in J_{r^{-1}r}$ and $i \in I_{ss^{-1}}$, we may define $\alpha_{(r,s)} \in P(I \times J, I)$ and $\beta_{(r,s)} \in P(I \times J, J)$ satisfying I by the expression $u_r(ji)u_s = (j, i)\alpha_{(r,s)}\, u_{rs}(j, i)\beta_{(r,s)}$, while, by applying the definitions of $\alpha_{(r,s)}$ and $\beta_{(r,s)}$ to $(u_s(jp))(u_t(qm)u_g) = (u_s(jp)u_t)((qm)u_g)$, we obtain II. Furthermore, since $(iu_sj)(pu_tq) = i(u_s(jp)u_t)q = i((j, p)\alpha_{(s,t)})\, u_{st}(j, p)\beta_{(s,t)}q$, $(iu_sj)\varphi = (i, s, j)$ defines an isomorphism of S onto (X, I, J, α, β). We next show that $T = (X, I, J, \alpha, \beta)$ is a regular semigroup such that $E(T)$ is a semigroup. We utilize I to establish closure and II to establish associativity. Utilizing I, $E(T) = \{(i, s, j) : s \in Y, i \in I_s, j \in J_s\}$.

and, hence, $E(T)$ is a semigroup since Y is a semigroup. If $(i, s, j) \in T$, $k \in I_{s^{-1}s}$, and $n \in J_{ss^{-1}}$, $(i, s, j)(k, s^{-1}, n)(i, s, j) = (i, s, j)$ by utilizing I.

The theorem may be specialized to give a structure theorem for idempotent semigroups. This topic will be treated separately [8].

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