SET-CONNECTED MAPPINGS

By Jin Ho Kwak

In this paper, we defined newly set-connected mappings and obtain a characterization and some properties about this mapping and find the relation between this mapping and a continuous mapping or a connected mapping. Furthermore, we shall find what conditions must be placed upon the spaces in order to be able to conclude that a set-connected mapping is continuous. For any concepts which we do not define or elaborate upon, the reader is referred to Dugundji's book [3].

DEFINITION 1. [4]. A space is said to be *connected between A and B* if there is no closed-open set F such that $A \subset F$ and $F \cap B = \phi$. Clearly, the connectedness between two sets is a symmetric relation.

DEFINITION 2. Let X and Y be topological spaces and $f: X \to Y$ be a mapping. If X is connected between A and B, then $f(X)$ is connected between $f(A)$ and $f(B)$ with respect to relative topology. Then f is called a *set-connected mapping* of X to Y.

LEMMA 1. [4]. *If a subspace of the space is connected between A and B, then so is the whole space.*

Now, we give a characterization of set-connected mappings.

THEOREM 2. *Let $f: X \to Y$ be a mapping. Then f is a set-connected mapping if and only if $f^{-1}(F)$ is a closed-open subset of X for any closed-open subset F of $f(X)$.*

PROOF. Only If; Let F be a closed-open subset of $f(X)$. Suppose $f^{-1}(F)$ is not closed in X. Let p be a limit point of f^{-1}(F) which does not belong to $f^{-1}(F)$. Then X is connected between p and $f^{-1}(F)$. Consequently, $f(X)$ is connected between $f(p)$ and $f[f^{-1}(F)]$, which contradicts to F is closed-open in $f(X)$ and $f[f^{-1}(F)] \subset F \subset \mathscr{C} f(p)$. Similarly, $f^{-1}(F)$ is a open subset of X.

If; Suppose $f(X)$ is not connected between $f(A)$ and $f(B)$. Then there exists a closed-open subset F in $f(X)$ such that $f(A) \subset F \subset \mathscr{C} f(B)$. By hypothesis, f^{-1}(F) is a closed-open subset of X and $A \subset f^{-1}(F) \subset \mathscr{C} B$. Therefore X is not connected between A and B.

REMARK. Also, if f is a set-connected, then $f^{-1}(F)$ is a closed-open subset of X for any closed-open subset F of Y.

COROLLARY 1. *Let $f : X \to Y$ be a set-connected mapping. If F is a closed-open subset of $f(X)$ (or Y), then each component of $f^{-1}(F)$ is closed in X.*

COROLLARY 2. *Every continuous mapping is a set-connected.*

Following example shows that the converse of corollary 2 is not true.

EXAMPLE 1. Let $f : E^1 \to E^1$ be defined by $f(x)=x^2$, $(x \neq 0)$ and $f(0)=1$, where E^1 is a Euclidean $1-$space. Then f is a set-connected mapping, (cf. lemma 3), but f is not continuous at 0.

REMARK. The set-connected mapping on a space to a 0-dimensional space is a continuous.

We may state the following lemma 3 and lemma 4 immediately.

LEMMA 3. *Every mapping f on X to Y such that $f(X)$ is a connected set is a set-connected mapping.*

LEMMA 4. *Let $f : X \to Y$ be a set-connected mapping. If X is a connected set, then $f(X)$ is a connected set.*

LEMMA 5. *Let $f : X \to Y$ be a set-connected mapping and A be a subset of X such that $f(A)$ is a closed-open subset of $f(X)$. Then $f|A : A \to Y$ is a set-connected.*

PROOF. Let A be connected between B_1 and B_2. By lemma 1, X is connected between B_1 and B_2. Therefore $f(X)$ is connected between $f(B_1)$ and $f(B_2)$.
Since $f(A)$ is a closed-open subset of $f(X)$, $f(A)$ is connected between $f(B_1)$ and $f(B_2)$.

THEOREM 6. *Let $f : X \to Y$ be a set-connected open surjection, and assume that each fiber $f^{-1}(y)$ is connected. Then for any closed-open subset F of Y, F is connected (component) if and only if $f^{-1}(F)$ is connected (component). In particular, Y is connected if and only if X is connected.*

PROOF. Only if; Suppose $f^{-1}(F)$ is not connected in X. Then there are open subsets A and B of X such that $f^{-1}(F) \cap A \cap B = \phi$, $f^{-1}(F) \subset A \cup B$ and $f^{-1}(F) \cap A \neq \phi \neq f^{-1}(F) \cap B$. Since $f^{-1}(y)$ is connected, either $f^{-1}(y) \subset A$ or $f^{-1}(y) \subset B$ for every $y \in F$. Therefore $F \cap f(A) \cap f(B) = \phi$, $F \subset f(A) \cup f(B)$ and $F \cap f(A) \neq \phi \neq F \cap f(B)$. Since f is open mapping, $f(A)$ and $f(B)$ are open subsets of

Y. Hence F is not connected.

If; Since $f[f^{-1}(F)]=F$ is closed-open in Y, by lemma 5 $f|f^{-1}(F)$ is a set-connected mapping and $f^{-1}(F)$ is a connected set. Hence by lemma 4, $f|f^{-1}(F)[f^{-1}(F)]=F$ is a connected set.

THEOREM 7. *Let $f: X \to Y$ be a set-connected mapping* and *the components of Y be open sets. Then for each $p \in Y$, $f[\overline{f^{-1}(p)}]$ is contained only one component. In fact, $f[\overline{f^{-1}(p)}]$ is contained the component containing p.*

PROOF. Let $C(p)$ denote the component containing p and $x \in \overline{f^{-1}(p)}$. Then X is connected between $f^{-1}(p)$ and x. Therefore Y is connected between p and $f(x)$. Since $C(p)$ is a closed-open subset of Y, $f(x)$ is contained in $C(p)$. Hence $f[\overline{f^{-1}(p)}] \subset C(p)$.

THEOREM 8. *Let X be a topological space and x has a connected neighborhood for each $x \in X$. If a sequence of points p_n of X converge to p, then there exist some n_0 such that X is connected between p and p_n for each $n \geq n_0$.*

PROOF. If p and p_n are contained a connected set, then X is connected between p and p_n. Hence we obtain the result easily.

COROLLARY 3. *If a sequence of points p_n of a locally connected space converge to p, then there exists some n_0 such that X is connected between p and p_n for each $n \geq n_0$.*

EXAMPLE 2. In above theorem 8, the hypothesis of "x has a connected neighborhood for each $x \in X$." is essential: in ordinal space $[0, \omega]$, where ω is the first infinite ordinal number, the sequence $0, 1, 2, \cdots\cdots, n, \cdots\cdots$ is converge to ω, but $[0, \omega]$ is not connected between ω and n for each $n \in [0, \omega[$.

In order to say the following, we note that a Hausdorff space in which the closure of every open sets is open is called an extremally disconnected space. [5].

THEOREM 9. *Let $f: X \to Y$ be a set-connected mapping and Y be an extremally disconnected space. Then $G(f)$ is closed in $X \times Y$.*

PROOF. Suppose $f(x) \neq y$. Then there exists a closed-open neighborhood V of y not containing $f(x)$, therefore $f^{-1}(V)$ is closed-open in X and $x \notin f^{-1}(V)$. Taking $U = X - f^{-1}(V)$ is a neighborhood of x, then $f(U) \cap V = \phi$. By Lemma 1 of [2], $G(f)$ is closed.

In [2], it was shown that any mapping on a first countable space into a countably compact space having a closed graph is continuous. Combining this with theorem 9, we have the result:

THEOREM 10. *Let $f: X \to Y$ be a set-connected mapping, X be a first countable space and let Y be an extremally disconnected countably compact space. Then f is continuous.*

THEOREM 11. *Let $f: X \to Y$ be a set-connected mapping, If x has a connected neighborhood for each $x \in X$ and Y is an extremally disconnected space, then f is continuous.*

PROOF. Let $x \in X$ be given. It is sufficient to show that $f(U)=f(x)$. for a connected neighborhood U of x. Suppose that there exist x' in U such that $f(x') \neq f(x)$. Then there is a closed-open subset V of Y such that $f(x) \in V$ and $f(x') \notin V$. Then $f^{-1}(V) \cap U$ is a nonempty closed-open proper subset of a subspace U, which contradicts.

Finally, the following example shows that the concepts: "set-connected mapping" and "connected mapping" are independent.

EXAMPLE 3. The mapping f defined in example 1 is a set-connected but not a connected. Next, let $X=[0, 1]-\left\{\frac{1}{n} \mid n \in Z^+\right\}$ be a subspace of E^1, where Z^+ is a set of all positive integers and let $Y=\{0\} \cup \left\{\frac{1}{n} \mid n \in Z^+\right\}$ be a space with discrete topology.

Define $g: X \to Y$ by $g(x)=\frac{1}{n+1}$ for any $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$, $(n \in Z^+)$ and $g(0)=0$. Then g is a connected mapping but not a set-connected.

Kyungpook University

REFERENCES

1. W. J. Pervin and N. Levine, *Connected mappings of Hausdorff space*, Proc. Amer. Math. Soc., Vol. 9 (1958), 488—495.
2. P. E. Long, *Functions with closed graphs*, Amer. Math. Monthly, Vol. 76 (1969), 930—932.
3. J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1968.
4. K. Kuratowski, *Topology* II, Academic Press, New York, 1968.
5. N. Bourbaki, *Elements of Mathematics*, General Topology I, Addison-Wesley, Reading, Mass., 1966.