ON A PROOF OF R. A. HUNT

By C. J. Mozzochi

Let $f \in L^1(-\pi, \pi)$. Let $S_n(x;f)$ be the n^{th} partial sum of the Fourier series for f. Let $M: L^p(-\pi, \pi) \to$ class of extended real valued functions on $(-\pi, \pi)$.

$$Mf(x) = \sup\{|S_n(x;f)| : |n| \geq 0\}$$

Let f^0 denote the 2π-periodic extension of f with domain $(-4\pi, 4\pi)$.

$$\text{Let } S_n^*(x; f; w_{-1}^*) = \text{P.V.} \int_{-4\pi}^{4\pi} \frac{e^{-int} f^0(t)}{x-t}\, dt \; ; \; |n| \geq 0.$$

Let $M^*: L^p(-\pi, \pi) \to$ class of extended real valued functions on $(-\pi, \pi)$.

$$M^* f(x) = \sup\{|S_n^*(x; f; w_{-1}^*)| : |n| \geq 0\}$$

On page 237 in [2] Hunt states without proof "It follows that (*) $\|M^* f\|_{p\infty} \leq A_p \|f\|_{p1}^*$ for all $f \in L(p, 1)$, $A_p \leq \text{const}\,(p/(p-1))B_p$, $1 < p < \infty$."

The only proof (produced later in this paper) of (*) that I know of is based on a delicate Lorentz space extension (communicated to me by C. Preston) of the very non-elementary theorem of M. Riesz which states that the Hilbert transform is of type (p, p) for $1 < p < \infty$.

In this paper (maintaining the use of his Lorentz space interpolation techniques together with only very elementary real variable techniques) I modify Hunt's proof in such a way as to eliminate (*) and the need for the theorem of M. Riesz to establish Theorem 1 in [2]. A similar modification will yield Theorem 2 and Theorem 3.

LEMMA 1. *$Mf(x) \leq E(\|f\|_p + M^* f(x))$ for almost every x in $(-\pi, \pi)$ for $1 < p < \infty$ where $E > 0$ is a constant independent of f in $L^1(-\pi, \pi)$.*

An exhaustive proof of this lemma is given in [3].

LEMMA 2. (Carleson-Hunt). *Let $F \subset (-\pi, \pi)$ and let χ_F be the characteristic function of F. For every $y > 0$ and $1 < p < \infty$ we have*

$$m\{x \in (-\pi, \pi) \mid M^* \chi_F(x) > y\} \leq B_p^p\, y^{-p} mF.$$

This is established in [2].

LEMMA 3. *Let $F \subset (-\pi, \pi)$ and let χ_F be the characteristic function of F. For every $y > 0$ and $1 < p < \infty$ we have*

$$m\{x\in(-\pi,\ \pi)\,|\,M\chi_F(x)>y\}\leq (1+2\pi)\ (2EB_p)^p\ y^{-p}\ mF$$

where E is of Lemma 1 *and B_p is of Lemma* 2.

PROOF. This follows from Lemma 1 and Lemma 2 in the following way:

$$M\chi_F(x)\leq E(M^*\chi_F(x)+(mF)^{1/p}) \text{ a.e. in } (-\pi,\pi).$$

$$m\{x\,|\,(mF)^{1/p}>y\}=\begin{cases}0 & y\geq(mF)^{1/p}\\ 2\pi & y<(mF)^{1/p}\end{cases}$$

$$m\{x\,|\,(mF)^{1/p}>y\}\leq 2\pi(1/mF)mF$$

$$m\{x\,|\,(mF)^{1/p}>y\}\leq 2\pi B_p^p\ y^{-p}mF$$

$$\{x\,|\,M^*\chi_F(x)+(mF)^{1/p}>y\}\subset\{x\,|\,M^*\chi_F(x)>y/2\}\cup\{x\,|\,(mF)^{1/p}>y/2\}$$

$$m\{x\,|\,M^*\chi_F(x)+(mF)^{1/p}>y\}\leq(1+2\pi)(2B_p)^p y^{-p}mF. \text{ Hence}$$

$$m\{x\,|\,M\chi_F(x)>y\}\leq(1+2\pi)(2EB_p)^p y^{-p}\ mF.$$

LEMMA 4. *Let $F\subset(-\pi,\ \pi)$ and let χ_F be the characteristic function of F. Then* $\|M\chi_F\|^*_{p\infty}\leq B_p'\|\chi_F\|^*_{p1}$, $1<p<\infty$ *where* $B_p'=(1+2\pi)^{1/p}(2EB_p)$

PROOF. This is an immediate consequence of Lemma 3 and the fact that $(mF)^{1/p}=\|\chi_F\|^*_{p1}$ and $\|M\chi_F\|^*_{p\infty}=\sup\{[\lambda_{M\chi_F}(y)]^{1/p}y\,|\,y>0\}$.

LEMMA 5. *For every simple function f in $L(p,1)$ for $1<p<\infty$*

$$\|Mf\|^*_{p\infty}\leq \text{Const}(p/p-1)\ B'_p\|f\|^*_{p1}$$

PROOF. This follows immediately from Lemma 4. For details see the argument on the bottom of page 236 in [2] (replace M^* by M in the argument).

Let α denote any simple function with domain $(-\pi,\ \pi)$ and range in $\{0,1,\cdots,N\}$. We say α is an Nth order simple function.

Let $T_\alpha f(x)=S_{\alpha(x)}(x;\ f)$ for $x\in(-\pi,\ \pi)$.

Clearly, T_α is linear for every Nth order simple function, and for $1<p<\infty$ $\|T_\alpha f\|_{p\infty}\leq C_p\|f\|^*_{p1}$ for every simple function f in $L(p,\ 1)$ and for each Nth order simple function α where $C_p=\text{Const}\ (p/p-1)B_p$

LEMMA 6. *Let $f\in L(p,\ 1)$ $1<p<\infty$. Let α be any Nth order simple function. There exists a sequence of simple functions $\{f_n\}\subset L(p,\ 1)$ such that $\|f_n\|_{p1}\to\|f\|_{p1}$ and $T_\alpha(f-f_n)(x)\to 0$ for $x\in(-\pi,\ \pi)$.*

PROOF. This is an immediate consequence of the Lebesgue dominated convergence theorem and (2.4) in [1].

LEMMA 7. *For* $1<p<\infty$ $\|T_\alpha f\|^*_{p\infty}\leq C'_p\|f\|^*_{p1}$ *for every* f *in* $L(p, 1)$ *and for each Nth order simple function* α.

PROOF. Fix $f\in L(p,1)$. Let $\{f_n\}$ be the sequence of Lemma 6. Then $|T_\alpha f_n(x)|\to|T_\alpha f(x)|$ for x in $(-\pi, \pi)$. Hence by Fatou's lemma we have $(T_\alpha f)^{**}(t)\leq\underline{\lim}(T_\alpha f_n)^{**}(t)$ and $\|T_\alpha f\|_{p\infty}\leq\underline{\lim}\|T_\alpha f_n\|_{p\infty}$. The result now follows from Lemma 5.

LEMMA 8. $\|Mf\|^*_{p\infty}\leq C''_p\|f\|^*_{p1}$ *for all* f *in* $L(p,1)$ $1<p<\infty$.

PROOF. Fix f_0 in $L(p,1)$. Clearly there exists an Nth order simple function α_0 such that $|T_{\alpha_0(x)}f_0(x)|=M_Nf_0(x)$ for all $x\in(-\pi,\pi)$. Hence $\|M_Nf_0\|_{p\infty}\leq C'_p\|f_0\|_{p1}$. But $M_Nf_0(x)\to Mf_0(x)$ for each x in $(-\pi, \pi)$; so that by Fatou's lemma $(Mf_0)^{**}(t)\leq\underline{\lim}(M_Nf_0)^{**}(t)$ and the result follows.

THEOREM 1. $\|Mf\|_p\leq K_p\|f\|_p$ $1<p<\infty$.

PROOF. This is an immediate consequence of Lemma 8 and the interpolation theorem found in [1].

To establish (*) we use the following Lorentz space generalization of M. Riesz theorem

$$\text{Let } h(x)=\int_{-4\pi}^{4\pi}\frac{f^0(t)}{x-t}\,dt \text{ for } x \text{ in } (-\pi,\pi).$$

LEMMA 9. $\|h\|_{p\infty}\leq A_p\|f\|^*_{p1}$ *for* $1<p<\infty$.

PROOF. By (1.8) in [1] it is sufficient to show $\binom{*}{*}$ $\|h\|_{p\infty}\leq A_p\|f\|_p$. Since the Hilbert transform is of type (p, p) it is of weak type (p, p). Hence $y[\lambda_h(y)]^{1/p}\leq A_p\|f\|_p$ for all $y>0$. Hence $\sup_{y>0} y\,[\lambda_h(y)]^{1/p}\leq A_p\|f\|_p$; so that by (1.7) in [1] $\binom{*}{*}$ follows.

PROOF of (*).

Let $T^*_\alpha f(x)=S^*_{\alpha(x)}(x;\, f;\, w_{-1})$ for x in $(-\pi, \pi)$. It is sufficient to establish

Lemma 6 for T_α^*. Clearly, for any Nth order simple function $\|T_\alpha^* f\|_{p\infty} \le \sum_0^N \|\int_{-4\pi}^{4\pi} \frac{e^{-\mathrm{int}} f^0(t)}{x-t}\, dt\|_{p\infty}$. But by Lemma 9 we have that the right side is majorized by $(N+1)A_p \|f\|_{p1}$. Let $\{f_n\} \subset L(p,1)$ be a sequence of simple functions such that $\|f_n - f\|_{p1} \to 0$. By (1.7) in [1] $T_\alpha^* f_n$ converges in measure to $T_\alpha^* f$. Hence there exists a sequence $\{f_{nk}\}$ such that almost everywhere $T^*(f-f_{n_k})(x) \to 0$.

REMARK 1. If f in $L \log L \log\log L$, then the Fourier series for f converges almost everywhere to f. This is due to L. Carleson, and it is based on Lemma 3. The proof is presented in detail for the Walsh orthogonal system on page 563—567 in [5]. It is easy to see that $L^1 \supset L \log L \log\log L \supset L^p$ for $1 < p$.

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