RELATIONS BETWEEN WIENER'S AND ROYDEN'S P-COMPACTIFICATIONS OF A RIEMANNIAN MANIFOLD

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Royden's P-compactification R^*_{MP} of a Riemannian manifold R was introduced by Nakai and Sario [6], and further investigated by Kwon, Sario, and Schiff [4], [5]. Wiener's P-compactification R^*_{NP} of a Riemann surface R was first discussed by Tanaka [10] and, more generally, Wiener's compactification of a harmonic space, by Constantinescu and Cornea [2]. In the present note we relate Wiener's and Royden's P-compactications of a Riemannian manifold by a continuous mapping ρ from R^*_{NP} onto R^*_{MP} which maps Wiener's P-harmonic boundary Δ_{NP} onto Royden's P-harmonic boundary Δ_{MP} except possibly for the P-singular point s_P. We show that the fiber of s_P is contained in Δ_{NP} if and only if R is P-hyperbolic. This observation has some interesting consequences: the P-elliptic measure of R has a finite energy integral if and only if $\int_R Pdv<\infty$; a P-hyperbolic Riemannian manifold R belongs to $\mathcal{O}_{PBE}$ if and only if $\Delta_{MP}=\{s_P\}$.

For fundamentals of the P-singular point we refer the reader to [6] and [3].

1. Let R be a noncompact Riemannian manifold of dimension $m\geq 2$. For a fixed nonnegative function $P\not\equiv 0$, we are interested in the class $P(R)$ of P-harmonic functions on R, that is, C^2 solutions of the elliptic partial differential equation

$$\Delta u = Pu.$$

Here Δ is the Laplace-Beltrami operator given by

$$\Delta u = -{*d*du}.$$

Let $\{\Omega\}$ be a regular exhaustion of R. The sequence $\{h_1^{P,\Omega}\}$ of nonnegative continuous functions on R with $h_1^{P,\Omega}=1$ on $R-\Omega$ and $h_1^{P,\Omega}\in P(\Omega)$ is monotone decreasing, and thus converges as $\Omega\to R$. The limit e_P is called the P-elliptic measure of R. A manifold R is, by definition, *P-parabolic* or *P-hyperbolic* according as $e_P\equiv 0$ or >0. The class of Riemannian manifolds on which there exist no bounded P-harmonic functions is identical with the class of P-parabolic manifolds (cf. [7] and [8]).

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2. A function on R is called *P-harmonizable* if for any regular exhaustion $\{\Omega\}$ of R, the sequence of the continuous functions $h_f^{P,\Omega}$ with $h_f^{P,\Omega}=f$ on $R-\Omega$ and $h_f^{P,\Omega}\in P(\Omega)$ converges as $\Omega\to R$. The limit h_f^P is P-harmonic. Denote by $N_P(R)$ the family of bounded continuous P-harmonizable functions on R, and by $N_{P\Delta}(R)$ the family of functions $f\in N_P(R)$ with $h_f^P\equiv 0$. Consider the N_P-compactification R^*_{NP} of R, defined by the following properties: R^*_{NP} is a compact Hausdorff space in which R is a dense subset; every $f\in N_P(R)$ has a continuous extension to R^*_{NP}; and $N_P(R)$ separates the points of R^*_{NP}. Such a compactification R^*_{NP} of R always exists and is unique up to a homeomorphism (e.g. [1]). In particular, R^*_{NP} can be chosen as the totality of nonzero multiplicative linear functionals on $N_P(R)$ with the weak * topology (e.g. [9]). We shall call R^*_{NP} *Wiener's P-compactification of* $R, R^*_{NP}-R$ *Wiener's P-boundary* of R, and the subset

$$\Delta_{NP}=\{x\in R^*_{NP}\,|\,f(x)=0 \text{ for all } f\in N_{P\Delta}(R)\}$$

Wiener's P-harmonic boundary.

The set Δ_{NP} enjoys the following properties related to the family of PB-functions:

(i) $R\in\mathcal{O}_{PB}$ if and only if $\Delta_{NP}=\phi$.

(ii) $N_P(R)=PB(R)\oplus N_{P\Delta}(R)$.

(iii) A PB-function on R takes on its nonnegative maximum and nonpositive minimum on Δ_{NP}.

(iv) The vector space $PB(R)$ is n-dimensional if and only if Δ_{NP} consists of n points.

The proofs of (i)-(iv) are analogous to those in the case of harmonic functions (cf. e.g. [9]).

3. Denote by $M_P(R)$ the family of bounded Tonelli functions f on R with finite energy integral

$$E_R(f)=\int_R df\wedge *df+\int_R Pf^2 *1,$$

and by $M_{P\Delta}(R)$ the family of BE-limits f of functions in $M_P(R)$ with compact supports, that is, $f=C\text{-}\lim_n f_n$ and $\lim_n E_R(f-f_n)=0$ on R, where $\{f_n\}$ is a sequence of uniformly bounded functions in $M_P(R)$ with compact supports, and C-lim stands for uniform convergence in compact sets. The M_P-compactification R^*_{MP} is called *Royden's P-compactification. Royden's P-harmonic boundary* is the subset

$$\triangle_{MP}=\{x\in R^*_{MP} \mid f(x)=0 \text{ for all } f\in M_{P\triangle}(R)\}.$$

A point s_P in R^*_{MP} is called a *P-singular point* if every function $f\in M_P(R)$ vanishes at s_P. It exists if and only if $\int_R Pdv=\infty$(cf. [6]). Since $M_P(R)$ separates the points in R^*_{MP}, it is unique. Clearly $s_P\in\triangle_{MP}$.

The P-singular point s_P will play an important role in our study.

By way of preparation we first observe:

PROPOSITION. *The following relations are valid*:

(i) $M_P(R)\subset N_P(R)$,

(ii) $M_{P\triangle}(R)=M_P(R)\cap N_{P\triangle}(R)$.

PROOF. Let $\{\Omega\}$ be a regular exhaustion of R, and $f\in M_P(R)$. Since the sequence $\{h_f^{P,\Omega}\}$ converges to h_f^P in the BE-topology, we have $f\in N_P(R)$. For every $f\in M_P(R)$,

$$h_f^P(x)=\int_{\triangle_{MP}} f(t)K(x,t)d\mu(t)$$

with $K(x,t)$ the P-harmonic kernel, and μ the P-harmonic measure with center at a fixed point (cf. [5]). Thus $f\in M_P(R)\cap N_{P\triangle}(R)$ if and only if $f\equiv 0$ on Δ_{MP}, and therefore if and only if $f\in M_{P\triangle}(R)$.

4. By the above proposition, every $f\in M_P(R)$ can be continuously extended to R^*_{NP}. Thus R^*_{NP} can be divided into equivalence classes of points by the following equivalence relation: $x\sim y$ if $f(x)=f(y)$ for all $f\in M_P(R)$. The quotient space $R^*_{NP}/\sim$ with the quotient topology is homeomorphic to Royden's P-compactification (cf. [3], [9]). In view of the natural projection ρ of R^*_{NP} onto the quotient space $R^*_{NP}/\sim$,

we have:

There exists a continuous mapping ρ of R^*_{NP} onto R^*_{MP} such that ρ is the identity mapping on R and $f(x)=f(\rho(x))$ for all $x\in R^*_{NP}$ and all $f\in M_P(R)$.

We shall refer to the inverse image of a point $x\in R^*_{MP}$ as the fiber of x.

5. Since $\triangle_{NP}$ and $\triangle_{MP}$ are closely related to the structure of PB and PBE respectively, the nature of the mapping ρ on $\triangle_{NP}$ is important. Since the P-singular point s_P does not affect the dimension of the vector space PBE, it is natural to ask whether or not the set $\rho^{-1}(s_P)$ belongs to $\triangle_{NP}$.

THEOREM 1. *The fiber of the P-singular point s_P is a subset of Wiener's P-harmonic boundary Δ_{NP} if and only if the base manifold R is P-hyperbolic.*

PROOF. Since the P-parabolicity of R implies $\Delta_{NP}=\phi$, the necessity is trivial. To prove the sufficiency, we assume to the contrary that $\rho^{-1}(s_P)\not\subset\Delta_{NP}$, that is, $s_P\notin\rho(\Delta_{NP})$. Since Δ_{NP} is compact and nonempty, so is $\rho(\Delta_{NP})$. One can find an $f\in M_P(R)$ with $f|\rho(\Delta_{NP})=1$. By the orthogonal decomposition $M_P(R)=PBE(R)\oplus M_{P\Delta}(R)$, the PBE-projection h_f^P of f has value 1 on $\rho(\Delta_{NP})$. Since $f(\rho(x))=f(x)$ for all $x\in R^*_{NP}$ and all $f\in M_P(R)$, $h_f^P|\Delta_{NP}=1$. In view of the maximum principle for PB-functions on Δ_{NP}, we see that h_f^P is the P-elliptic measure e_P of R.

Thus $E_R(e_P)<\infty$ and

$$\int_R P(e_P)^2 dv=\int_R Pe_P \lim_{\Omega\to R} h_1^{P,\Omega}\, dv <\infty.$$

By the monotone convergence theorem,

$$\lim_{\Omega\to R}\int_R Pe_P h_1^{P,\Omega}\, dv <\infty.$$

Choose a regular subregion Ω_0 such that

$$\int_R Pe_P h_1^{P,\Omega_0} dv <\infty.$$

Since Pe_P is nonnegative and $h_1^{P,\Omega_0}=1$ on $R-\Omega_0$,

$$\int_{R-\Omega_0} Pe_P dv <\infty.$$

By the continuity of Pe_P, $\int_{\Omega_0} Pe_P dv<\infty$, and therefore

$$\int_R Pe_P\, dv <\infty.$$

By repeating the same argument we obtain

$$\int_R P\, dv <\infty.$$

This implies that the P-singular point s_P does not exist, a contradiction.

6. **Theorem 1 has the following consequences.**

THEOREM 2. *On a P-hyperbolic manifold R, the following relations are equivalent:*

(i) $D_R(e_P)+\int_R P(e_P)^2\, dv<\infty$,

(ii) $\int_R P(e_P)^2\, dv <\infty$,

(iii) $\int_R P dv <\infty$.

PROOF. (i)⇒(ii) is trivial and (ii)⇒(iii) follows from the proof of Theorem 1. For (iii)⇒(i) we refer to [8].

COROLLARY. *If* $\int_R P(e_P)^2 dv < \infty$, *then* $D_R(e_P) < \infty$.

THEOREM 3. *On a P-hyperbolic Riemannian manifold R,* $\rho(\Delta_{NP}) = \Delta_{MP}$.

PROOF. Since $M_{P\Delta}(R) \subset N_{P\Delta}(R)$, we have $\rho(\Delta_{NP}) \subset \Delta_{MP}$ by the definitions of P-harmonic boundaries and the mapping ρ. It remains to show that $\Delta_{MP} - \rho(\Delta_{NP}) = \phi$. Suppose $\Delta_{MP} - \rho(\Delta_{NP}) \neq \phi$. Since $s_P \in \rho(\Delta_{NP})$ by Theorem 1, one can find a point $x \in \Delta_{MP} - \rho(\Delta_{NP})$ and a function $f \in M_P(R)$ such that $0 \leq f \leq 1, f(x) = 1$, and $f \equiv 0$ on $\rho(\Delta_{NP})$ (cf. [4]). Thus the PBE-projection u of f coincides with f on Δ_{MP}. Since $u(x) = f(x) = 1, u > 0$ on R. On the other hand $u(\Delta_{NP}) = u(\rho(\Delta_{NP})) = 0$, and therefore $u \equiv 0$ on R, a contradiction.

Note that the P-singular point s_P does not exist if $\int_R P dv < \infty$.

COROLLARY 1. *If* $\int_R P dv < \infty$, *then* $\rho(\Delta_{NP}) = \Delta_{MP}$.

COROLLARY 2. *If* $R \notin \mathcal{O}_{PB}$ *and* $\int_R P dv = \infty$, *then the dimension of the vector space* $PBE(R)$ *is strictly less than the dimension of* $PB(R)$.

PROOF. If $R \notin \mathcal{O}_{PB}$ and $\int_R P dv = \infty$, then $e_P \in PB(R)$ and $e_P \notin PBE(R)$.

COROLLARY 3. *If* $R \notin \mathcal{O}_{PB}$, *then* $R \in \mathcal{O}_{PBE}$ *if and only if* $\Delta_{MP} = \{s_P\}$.

COROLLARY 4. *A manifold* $R \in \mathcal{O}_{PBE} - \mathcal{O}_{PB}$ *if and only if* $\Delta_{MP} = \rho(\Delta_{NP}) = \{s_P\}$.

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