ESTIMATION OF SYMMETRIC OPERATORS IN BANACH SPACES

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The purpose of this note is to extend the trivial fact in the real line, R, that if $f: R \to R$ and constants $a, b > 0$ such that

$f(y)z = o(z^2)$ with $a|y| < |z| < b|y|$, then $\lim_{y \to 0} \frac{|f(y)|}{|y|} = 0$.

The extension is a useful estimation in the calculus on Banach spaces as for example in the proof of the converse of Taylor's Theorem. [1] [2]

Let U be an open neighborhood of the origin in a Banach space E and let $f: U \to L_s^k(E, F)$ be a map from U into the space of bounded, symmetric k-linear maps of E into a Banach space F. $L_s^k(E, F)$ is a closed subspace of the Banach space $L^k(E, F)$ of bounded, k-linear maps of E into F, with the operator norm induced from that of E and F, i.e., if $A \in L^k(E, F)$, then

$$\|A\| = \sup\{\|A(h_1, \cdots\cdots, h_k)\|_F \mid h_i \in E, \ |h_i| \leq 1\}.$$

We note that there is a norm-preserving isomorphism of $L^k(E, F)$ onto $L(E, \cdots\cdots, L(E, L(E, F))\cdots)$, k times, under the identification which takes an element of the latter into the form given by

$$A(y_1, \cdots\cdots, y_n) = ((\cdots(Ay_1)\cdots\cdots)y_{n-1})y_n.$$

If A is in $L^k(E, F)$ we denote the value $A(y_1, \cdots\cdots, y_n)$ by $Ay_1 \cdots y_n$. Also if y is in E, the y^n means $(y, \cdots\cdots, y)$, n times. Hence

$$Az_1^{n_1} \cdots z_i^{n_i} = (\cdots(Az_1^{n_1}) \ z_2^{n_2})\cdots)z_i^{n_i} = A(z_1, \cdots, \cdots z_1, \cdots, z_i, \cdots, z_i).$$

We adopt the following standard notation:

$$Az^k = o(z^{k+\gamma}) \text{ where } \gamma > 0, \text{ means } \lim_{z \to 0} \frac{|Az^k|}{|z|^{k+\gamma}} = 0.$$

THEOREM. *Let γ be a non-negative real number. Let E and F be Banach spaces and U an open subset of E. Let $f: U \to L_s^k(E, F)$. If there exist constants $a, b > 0$ such that $f(y)z^k = o(z^{k+\gamma})$ for $a|y| < |z| < b|y|$ for $z, y \in E$, then $f(y) = o(y^{\gamma})$.*

PROOF. By induction on k. For $k=1$, we have $f(y)z = o(z^{1+\gamma})$. Given $\varepsilon > 0$, $\delta > 0$ such that

$$\frac{|f(y)z|}{|z|^{1+\gamma}} < \frac{\varepsilon}{b^{\gamma}} \quad \text{for } |z| < b\delta.$$

This implies

$$\left| f(y)\frac{z}{|z|} \right| < \frac{\varepsilon}{b^{\gamma}}|z|^{\gamma} < \varepsilon\,|y|^{\gamma} \text{ since } |z|<b|y|, \text{ for } |y|<\delta.$$

Taking supremum over $z \leq \frac{b\delta}{2}$ we obtain

$$\|f(y)\| < \varepsilon |y|^{\gamma} \text{ for } |y|<\delta.$$

Thus $f(y)=o(y^{\gamma})$.

Assume that the theorem is true for $k<n$. We first note that we can choose $c, d>0$ $a<c<d<b$, and $a|y|<|z_1+\lambda z_2|<b|y|$ for z_1 and z_2 with $c|y|<|z_i|<d|y|$, $i=1, 2$, provided λ is small enough, as it easily follows from

$$|z_1|-|\lambda||z_2| \leq |z_1+\lambda z_2| \leq |z_1|+|\lambda||z_2|.$$

This enables us to substitute freely the element in o (element) estimates.

By multilinearity,

$$\begin{aligned} f(y)(z_1+\lambda_i z_2)^n &= f(y)z_1^n + \lambda_i C_1^n f(y) z_1^{n-1} z_2 + \cdots\cdots + \\ &\quad \lambda_i^{n-1} C_{n-1}^n f(y) z_1 z_2^{n-1} + \lambda_i^n f(y) z_2^n \\ &= o(y^{n+\gamma}). \end{aligned}$$

Since $f(y)z_1^n = o(y^{n+\gamma})$ and $f(y)z_2^n = o(y^{n+\gamma})$, we have

$$\lambda_i C_1^n f(y) z_1^{n-1} z_2 + \cdots\cdots + \lambda_i^{n-1} C_{n-1}^n z_1 z_2^{n-1} = o(y^{n+1}).$$

We now choose $\lambda_i \neq \lambda_j$ for $i \neq j$, $i=1, 2, \cdots\cdots, n-1$, and each λ_i sufficiently small. Thus we have a system of equations in matrix form

$$\begin{pmatrix} \lambda_1 & \lambda_1^2 & \cdots\cdots & \lambda_1^{n-1} \\ \lambda_2 & \lambda_2^2 & \cdots\cdots & \lambda_2^{n-1} \\ \vdots & & & \vdots \\ \lambda_{n-1} & \lambda_{n-1}^2 & \cdots\cdots & \lambda_{n-1}^{n-1} \end{pmatrix} \begin{pmatrix} C_1^n f(y) z_1^{n-1} z_2 \\ C_2^n f(y) z_1^{n-2} z_2^2 \\ \vdots \\ C_{n-1}^n f(y) z_1 z_2^{n-1} \end{pmatrix} = \begin{pmatrix} o(y^{n+\gamma}) \\ o(y^{n+\gamma}) \\ \vdots \\ o(y^{n+\gamma}) \end{pmatrix}$$

Since $\lambda_i \neq \lambda_j$, the above matrix is invertible. Hence $f(y)z_1 z_2^{n-1} = o(y^{n+\gamma})$.

$$\left(f(y)\frac{z_1}{|z_1|} \right) z_2^{n-1} = o(y^{n-1+\gamma}).$$

By induction hypothesis, $f(y)\frac{z_1}{|z_1|} = o(y^{\gamma})$. Again by case $k=1$, $f(y)=o(y^{\gamma})$.

This concludes the proof.

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REFERENCES

[1] R. Abraham and J. Robbin, *Transversal Mappings and Flows*, W.A. Benjamin, 1967.

[2] E. Nelson, *Topics in Dynamics, I: Flows*, Math. Notes, Princeton University Press, 1969.