AN IDENTITY IN COMBINATIONS

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If we have $k(k\geq 1)$ objects $a_1, a_2, \cdots\cdots, a_k$, then there exist

(1) $$E(k, n)=\sum_{j=0}^{r}(-1)^j\binom{r}{j}(k-j)^n$$

different permutations of the k objects taken $n(n>0)$ at a time where repetitions are permitted and which contains each of $a_1, a_2, \cdots\cdots, a_r$ $(r\leq k)$ at least once. This result does not appear to have been explicitly stated or any rate does not seem well known.

In this note we shall establish (1) by using the well-known principle of inclusion and exclusion [4, pp. 233] and [6, pp. 19]. In general notation follows [6].

Let S be a finite set of n elements. When this terminology is used we assume $n>0$ and exclude the empty set ϕ. Let each a in S be assigned a unique weight $W(a)$. Let P denote an k-set of k properties

(2) $$P_1, P_2, \cdots\cdots, P_k$$

connected with the elements of S and let

(3) $$\{P_{i_1}, P_{i_2}, \cdots\cdots, P_{i_r}\} \ (r\leq k)$$

denote an r-subset of P.

Let

(4) $$W(P_{i_1}, P_{i_2}, \cdots\cdots, P_{i_r})$$

equal the sum of the weights of those elements of S that satisfy each of the properties $P_{i_1}, P_{i_2}, \cdots\cdots, P_{i_r}$. Now let

(5) $$W(r)=\sum W(P_{i_1}, P_{i_2}, \cdots\cdots, P_{i_r})$$

equal the sum of the quantities (4) over all of the r-subsets of P. Let $E(0)$ denote the sum of the weights of the elements of S that satisfy none of the properties (2). Then

(6) $$E(0)=W(0)-W(1)+W(2)-\cdots+(-1)^k W(k).$$

Formula (6) is known as the sieve formula, and proved by Hardy and Wrigth [4, pp. 233] and Ryser [6, pp. 19].

In order to establish (1) we let $P_j(1\leq j\leq r)$ be the number of n-tuples of objects chosen from $a_1, a_2, \cdots\cdots, a_k$ which do not contain a_j (viz $(k-j)^n$). Then

$$W(P_{i_1}, P_{i_2}, \cdots\cdots, P_{i_r})=(k-j)^n$$

and

$$W(j)=\binom{r}{j}(k-j)^n$$

Hence by (6) we obtain the following formula

(7) $$E(k,\ n)=\sum_{j=0}^{r}(-1)^j\binom{r}{j}(k-j)^n=\begin{cases}0 & \text{if } 1\leq n<r,\\ r! & \text{if } n=r.\end{cases}$$

Hence (7) is the number of n-tuples of k-objects, repetitions permitted, which contains $a_1,\ a_2,\ \cdots,\ a_r(r\leq k)$ at least once. In (7), if $r=k$, then we derive the well-known formula

(8) $$E(k,\ n)=\sum_{j=0}^{r}(-1)^j\binom{k}{j}(k-j)^n=\begin{cases}0 & \text{if } 1\leq n<k,\\ k\,! & \text{if } n=k.\end{cases}$$

Formula (8) is also the number of ordered partions of a finite set of n objects into k disjoint non-empty subsets.

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