A NOTE ON GENERALIZED MEIJER FUNCTIONS OF TWO VARIABLES

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In this note the author evaluates two integrals involving generalized Meijer functions of two variables with Gauss's hypergeometric functions which are generalizations of the results recently given by Pathak [(6)]. Some interesting particular cases and double-integral analogues of the results are also obtained.

1. Introduction.

A generalization of Meijer's G-function [(4), p. 207, (1)]

$$(1.1)\quad G^{m,n}_{p,q}\left[x\,\middle|\,{a_p \atop b_q}\right]=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}x^s ds$$

has been introduced by Sharma [(7), pp. 26—40] in the form

$$(1.2)\quad S\left[{x \atop y}\right]\equiv S\left[\begin{matrix}\left[\begin{matrix}p, & 0\\ A-p, & B\end{matrix}\right]\\ \left(\begin{matrix}q, & r\\ C-q, & D-r\end{matrix}\right)\\ \left(\begin{matrix}k, & l\\ E-k, & F-l\end{matrix}\right)\end{matrix}\,\middle|\,\begin{matrix}(a):(b)\\ (c):(d)\\ (e):(f)\end{matrix}\,\middle|\,\begin{matrix}x\\ y\end{matrix}\right]=\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Phi(s+t)\Psi(s,t)x^s y^t ds dt$$

where L_1, L_2 are suitable contours and

$$\Phi(s+t)=\frac{\prod_{j=1}^{p}\Gamma(a_j+s+t)}{\prod_{j=p+1}^{A}\Gamma(1-a_j-s-t)\prod_{j=1}^{B}\Gamma(b_j+s+t)},$$

$$\Psi(s,t)=\frac{\prod_{j=1}^{q}\Gamma(1-c_j+s)\prod_{j=1}^{r}\Gamma(d_j-s)\prod_{j=1}^{k}\Gamma(1-e_j+t)\prod_{j=1}^{l}\Gamma(f_j-t)}{\prod_{j=q+1}^{C}\Gamma(c_j-s)\prod_{j=r+1}^{D}\Gamma(1-d_j+s)\prod_{j=k+1}^{E}\Gamma(e_j-t)\prod_{j=l+1}^{F}\Gamma(1-f_j+t)}.$$

A, B, C etc. are positive integers and satisfy the following inequalities: $D\geqslant 1$, $F\geqslant 1$, $A\geqslant 0$, $B\geqslant 0$, $0\leqslant p\leqslant A$, $0\leqslant q\leqslant C$, $0\leqslant r\leqslant D$, $0\leqslant k\leqslant E$, $0\leqslant l\leqslant F$, $A+C\leqslant B+D$ and $A+E\leqslant B+F$.

The integral (1.2) converges if

$$(1.3)\quad \begin{cases}2(p+q+r)>A+B+C+D, & |\arg(x)|<[p+q+r-\frac{1}{2}(A+B+C+D)]\pi,\\ 2(p+k+l)>A+B+E+F, & |\arg(y)|<[p+k+l-\frac{1}{2}(A+B+E+F)]\pi\\ & \text{or}\end{cases}$$

$$\begin{cases} A+C<B+D,\ A+E<B+F, \\ \text{or else } A+C=B+D,\ A+E=B+F \text{ with } |x|<1,\ |y|<1. \end{cases}$$

To economise space, (a) stands for the set of A-parameters $a_1, a_2, \cdots\cdots, a_p;$ $a_{p+1}, \cdots, a_A$ with similar interpretation for (b), (c), (d), (e), (f) and $a_p(b_q)$ for $p(q)$ parameters $a_1, \cdots\cdots, a_p(b_1, \cdots\cdots, b_q)$.

Agarwal [(2), p. 587] has also given an extension of Meijer's G-function in two variables with slight difference in the parameters in the form.

The aim of this note is to evaluate some finite integrals for the product of generalized Meijer functions of two variables with Gauss's hypergeometric functions which generalize certain known results obtained by Pathak [(6)] and also discuss their double integral analogues. Since the generalized Meijer function of two variables is a very general function, several integrals involving the product of two G-functions, Kampé de Fériet's hypergeometric function which, in turn, yields Appell's functions [(1)] F_1, F_2, F_3 and F_4, Whittaker function of two variables etc., with Gauss's hypergeometric functions follow as particular cases of our findings.

We shall require the following formulae and relations in the present investigation.

(a) Integral [(5), p. 850, (12)] :

$$(1.4)\quad \int_0^1 x^{\nu-1}(1-x)^{\mu-1}{}_pF_q\left(\begin{matrix}a_p\\b_q\end{matrix}\Big|x\right)dx=\frac{\Gamma(\nu)\Gamma(\mu)}{\Gamma(\nu+\mu)}{}_{p+1}F_{q+1}\left(\begin{matrix}a_p, & \nu\\ b_q, & \nu+\mu\end{matrix}\Big|\alpha\right),$$

where Re $(\nu)>0$, Re $(\mu)>0$, $p\leqslant q+1$, if $p=q+1$, then $|\alpha|<1$.

(b) Watson's theorem [(4), p. 189, (6)] :

$$(1.5)\quad {}_3F_2\left(\begin{matrix}a, \ b, \ c\\ \frac{1}{2}(a+b+1), \ 2c\end{matrix};\right)$$

$$=\frac{\sqrt{\pi}\,\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}-\frac{1}{2}a\right)\Gamma\left(c+\frac{1}{2}-\frac{1}{2}b\right)}$$

where $\mathrm{Re}(c)>-\frac{1}{2}$, $\mathrm{Re}(a+b)>-1$, $\mathrm{Re}(2c-a-b)>-1$.

(c) Whipple's theorem [(4), p. 189, (7)] :

$$(1.6)\quad {}_3F_2\left(\begin{matrix}a, \ 1-a, \ c\\ f, \ 2c+1-f\end{matrix};\right)$$

$$=\frac{\pi\Gamma(f)\Gamma(2c+1-f)2^{1-2c}}{\Gamma\left(c+\frac{1}{2}a+\frac{1}{2}-\frac{1}{2}f\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}f\right)\Gamma\left(c+1-\frac{1}{2}a-\frac{1}{2}f\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}a+\frac{1}{2}f\right)}$$

where $\mathrm{Re}(f)>0$ and $\mathrm{Re}(2c-f)>-1$.

(d) The multiplication formula for the Gamma function [(4), p.4, (11)] :

$$(1.7)\quad \Gamma(mz)=(2\pi)^{\frac{1}{2}-\frac{1}{2}m}m^{mz-\frac{1}{2}}\prod_{i=0}^{m-1}\Gamma\left(z+\frac{i}{m}\right),\ m=2,\ 3,\ 4,\ \cdots\cdots$$

(e) Kampé de Fériet's function [(1), p.150] denoted in a slightly modified notation as

$$(1.8)\quad F^{m,\ l}_{n,\ p}\begin{bmatrix}x\\y\end{bmatrix}=F^{m,\ l}_{n,\ p}\left[\begin{matrix}a_m : b_l : b'_l\\ c_n : d_p : d'_p\end{matrix}\middle|\begin{matrix}x\\y\end{matrix}\right]$$

2. The general integrals.

The first integral to be evaluated is:

$$(2.1)\quad \int_0^1 x^{\rho-1}(1-x)^{\rho-1}{}_2F_1\left(\begin{matrix}\alpha,\ \beta\\ \frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\end{matrix};x\right)S\begin{bmatrix}ux^{\lambda}(1-x)^{\lambda}\\ vx^{\lambda}(1-x)^{\lambda}\end{bmatrix}dx$$

$$=\frac{\pi\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\right)}{\lambda^{1/2}2^{2\rho-1}\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)}$$

$$\times S\left[\begin{matrix}\begin{bmatrix}p+2\lambda, & 0\\ A-p, & B+2\lambda\end{bmatrix}\\ \begin{pmatrix}q, & r\\ C-q, & D-r\end{pmatrix}\\ \begin{pmatrix}k, & l\\ E-k, & F-l\end{pmatrix}\end{matrix}\middle|\begin{matrix}\Delta(\lambda,\rho),\ \Delta\left(\lambda,\ \rho+\frac{1}{2}-\frac{1}{2}\alpha-\frac{1}{2}\beta\right),\ (a):\\ \Delta\left(\lambda,\ \rho+\frac{1}{2}-\frac{1}{2}\alpha\right),\ \Delta\left(\lambda,\rho+\frac{1}{2}-\frac{1}{2}\beta\right),\ (b)\\ (c):(d)\\ (e):(f)\end{matrix}\middle|\begin{matrix}u/2^{2\lambda}\\ \\ v/2^{2\lambda}\end{matrix}\right]$$

where λ is a positive integer, $\mathrm{Re}(\alpha+\beta)>-1$ and

$$\text{(i)}\quad\begin{cases}2(p+q+r)>A+B+C+D,\ \ |\arg(u)|<\left[p+q+r-\frac{1}{2}(A+B+C+D)\right]\pi,\\ 2(p+k+l)>A+B+E+F,\ \ |\arg(v)|<\left[p+k+l-\frac{1}{2}(A+B+E+F)\right]\pi,\\ \mathrm{Re}(\rho+\lambda d_{h_1}+\lambda f_{h_2})>0,\ \ (h_1=1,\ 2,\ \cdots,\ r;h_2=1,\ 2,\ \cdots,\ l),\end{cases}$$

or

$$\text{(ii)}\quad\begin{cases}A+C<B+D,\ A+E<B+F,\\ \text{or else } A+C=B+D,\ A+E=B+F \text{ with } |u|<1,\ |v|<1,\\ \mathrm{Re}(\rho+\lambda d_{h_1}+\lambda f_{h_2})>0,\ (h_1=1,\ 2,\ \cdots,\ r;h_2=1,\ 2,\ \cdots,\ l).\end{cases}$$

The symbol $\Delta(m,n)$ stands for m-parameters:

$$\frac{n}{m},\ \frac{n+1}{m},\ \cdots\cdots,\ \frac{n+m-1}{m}.$$

The second integral to be evaluated is:

$$(2.2)\quad \int_0^1 x^{\rho-1}(1-x)^{\rho-\beta}{}_2F_1\left(\begin{matrix}\alpha,\ 1-\alpha\\ \beta\end{matrix};x\right)S\left[\begin{matrix}ux^{\lambda}(1-x)^{\lambda}\\ vx^{\lambda}(1-x)^{\lambda}\end{matrix}\right]dx$$

$$=\frac{\pi\,\Gamma(\beta)2^{1-2\rho}}{\sqrt{\lambda}\,\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\alpha+\frac{1}{2}\beta\right)}$$

$$\times S\left[\begin{matrix}\left[\begin{matrix}p+2\lambda,\ 0\\ A-p,\ B+2\lambda\end{matrix}\right]\\ \left(\begin{matrix}q\ ,r\\ C-q,\ D-r\end{matrix}\right)\\ \left(\begin{matrix}k,\ l\\ E-k,\ F-l\end{matrix}\right)\end{matrix}\middle|\begin{matrix}\Delta(\lambda,\rho),\ \Delta(\lambda,\rho-\beta+1),\ (a):\\ \Delta\left(\lambda,\rho+\frac{1}{2}\alpha+\frac{1}{2}-\frac{1}{2}\beta\right),\\ \Delta\left(\lambda,\rho+1-\frac{1}{2}\alpha-\frac{1}{2}\beta\right),\ (b)\\ (c):(d)\\ (e):(f)\end{matrix}\middle|\begin{matrix}\frac{u}{2^{2\lambda}}\\ \frac{v}{2^{2\lambda}}\end{matrix}\right]$$

where λ is a positive integer, $\mathrm{Re}(\beta)>0$ and converges if

$$\text{(i)}\ \begin{cases}2(p+q+r)>A+B+C+D,\quad |\arg(u)|<\left[p+q+r-\frac{1}{2}(A+B+C+D)\right]\pi,\\ 2(p+k+l)>A+B+E+F,\quad |\arg(v)|<\left[p+k+l-\frac{1}{2}(A+B+E+F)\right],\\ \mathrm{Re}(\rho+\lambda d_{h_1}+\lambda f_{h_2})>0,\ \mathrm{Re}(\rho-\beta+\lambda d_{h_1}+\lambda f_{h_2})>-1,\\ (h_1=1,\ 2,\ \cdots,\ r;h_2=1,\ 2,\ \cdots,\ l)\end{cases}$$

or

$$\text{(ii)}\ \begin{cases}A+C<B+D,\ A+E<B+F,\\ \text{or else } A+C=B+D,\ A+E=B+F \text{ with } |u|<1,\ |v|<1,\\ \mathrm{Re}(\rho+\lambda d_{h_1}+\lambda f_{h_2})>0,\ \mathrm{Re}(\rho-\beta+\lambda d_{h_1}+\lambda f_{h_2})>-1,\\ (h_1=1,\ 2,\ \cdots,\ r\ ;h_2=1,\ 2,\ \cdots,\ l).\end{cases}$$

Proof of (2.1):

Substituting the value of $S\left[\begin{matrix}ux^{\lambda}(1-x)^{\lambda}\\ vx^{\lambda}(1-x)^{\lambda}\end{matrix}\right]$ from (1.2) in the integrand of (2.1), changing the order of integration, and then evaluating the x-integral by making use of (1.4), we obtain

$$(2.3)\quad \frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Phi(s+t)\Psi(s,t)\frac{\Gamma(\rho+\lambda s+\lambda t)\Gamma(\rho+\lambda s+\lambda t)}{\Gamma(2\rho+2\lambda s+2\lambda t)}$$

$$\times{}_3F_2\left(\begin{matrix}\alpha,\ \beta,\ \rho+\lambda s+\lambda t\\ \frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2},\ 2\rho+2\lambda s+2\lambda t\end{matrix};\right)u^s v^t\,dsdt.$$

The change in order of integration involved here is easily justified by the application of de la Vallée Poussin's theorem [(3), p. 504], under the conditions referred to earlier in (2.1).

Now applying (1.5) in (2.3), then by virtue of (1.7), the expression becomes

$$(2.4)\quad \frac{\pi\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\right)2^{1-2\rho}}{\sqrt{\lambda}\,\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)}\ \frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Phi(s+t)\,\Psi(s,t)$$

$$\times\frac{\prod_{i=0}^{\lambda-1}\Gamma\left(\frac{\rho+i}{\lambda}+s+t\right)\prod_{i=0}^{\lambda-i}\Gamma\left(\frac{\rho+\frac{1}{2}-\frac{1}{2}\alpha-\frac{1}{2}\beta+i}{\lambda}+s+t\right)u^s v^t}{\prod_{i=0}^{\lambda-1}\Gamma\left(\frac{\rho+\frac{1}{2}-\frac{1}{2}\alpha+i}{\lambda}+s+t\right)\prod_{i=0}^{\lambda-1}\Gamma\left(\frac{\rho+\frac{1}{2}-\frac{1}{2}\beta+i}{\lambda}+s+t\right)}\,dsdt$$

where the contour L_1 is in the s-plane and runs from $-i\infty$ to $+i\infty$ with loops, if necessary to ensure that the poles of $\Gamma(d_j-s)$, $(j=1, 2, \cdots, r)$ lie to the right and the poles of $\Gamma(1-c_j+s)$, $(j=1, 2, \cdots, q)$ $\Gamma(a_j+s+t)$, $(j=1, 2, \cdots, p)$ and $\Gamma\left(\frac{\rho+i}{\lambda}+s+t\right)$, $\Gamma\left(\frac{\rho+\frac{1}{2}-\frac{1}{2}\alpha-\frac{1}{2}\beta+i}{\lambda}+s+t\right)$, $\{i=0, 1, \cdots, (\lambda-1)\}$ to the left of the contour.

Similarly the contour L_2 is in the t-plane and runs from $-i\infty$ to $+i\infty$ with loops, if necessary to ensure that the poles of $\Gamma(f_j-t)$, $(j=1, 2, \cdots, l)$ lie to the right and the poles of $\Gamma(1-e_j+t)$, $(j=1, 2, \cdots, k)$, $\Gamma(a_j+s+t)$, $(j=1, 2, \cdots, p)$ and $\Gamma\left(\frac{\rho+i}{\lambda}+s+t\right)$, $\Gamma\left(\frac{\rho+\frac{1}{2}-\frac{1}{2}\alpha-\frac{1}{2}\beta+i}{\lambda}+s+t\right)$, $\{i=0, 1, \cdots, (\lambda-1)\}$ to the left of the contour.

Therefore, on interpreting (2.4) in view of (1.2), we obtain the value of the integral (2.1).

The proof of integral (2.2) is similar to (2.1) except using (1.6) in place of (1.5).

3. Particular cases.

(a) On taking $A=B=p=0$ in (2.1), we get

$$(3.1)\quad \int_0^1 x^{\rho-1}(1-x)^{\rho-1}{}_2F_1\left(\begin{matrix}\alpha,\ \beta\\ \frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\end{matrix};\ x\right)\ G^{r,\ q}_{C,\ D}\left[ux^\lambda(1-x)^\lambda\,\middle|\,\begin{matrix}(c)\\(d)\end{matrix}\right]$$

$$\times G^{l,\ k}_{E,\ F}\left[vx^\lambda(1-x)^\lambda\,\middle|\,\begin{matrix}(e)\\(f)\end{matrix}\right]dx$$

$$=\frac{\pi 2^{1-2\rho}\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\right)}{\sqrt{\lambda}\,\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)}$$

$$\times S\left[\begin{matrix}\begin{bmatrix}2\lambda, & 0\\ 0, & 2\lambda\end{bmatrix}\\ \begin{pmatrix}q, & r\\ C-q, & D-r\end{pmatrix}\\ \begin{pmatrix}k, & l\\ E-k, & F-l\end{pmatrix}\end{matrix}\left|\begin{matrix}\Delta(\lambda,\rho)\ \Delta\left(\lambda,\ \rho+\frac{1}{2}-\frac{1}{2}\alpha-\frac{1}{2}\beta\right):\\ \Delta\left(\lambda,\ \rho+\frac{1}{2}-\frac{1}{2}\alpha\right),\ \Delta\left(\lambda,\ \rho+\frac{1}{2}-\frac{1}{2}\beta\right)\\ (c):(d)\\ (e):(f)\end{matrix}\right|\begin{matrix}\frac{u}{2^{2\lambda}}\\ \frac{v}{2^{2\lambda}}\end{matrix}\right]$$

where λ is a positive integer, $\mathrm{Re}(\alpha+\beta)>-1$ and

(i) $\begin{cases}2(p+r)>C+D,\ |\arg(u)|<\left[q+r-\frac{1}{2}(C+D)\right]\pi,\\ 2(k+l)>E+F,\ |\arg(v)|<\left[k+l-\frac{1}{2}(E+F)\right]\pi\\ \mathrm{Re}(\rho+\lambda d_{h_1}+\lambda f_{h_2})>0,\ (h_1=1,\ 2,\ \cdots,\ r;h_2=1,\ 2,\ \cdots,\ l)\end{cases}$

or

(ii) $\begin{cases}C<D,\ E<F, \text{ or else } C=D,\ E=F \text{ with } |u|<1,\ |v|<1,\\ \mathrm{Re}(\rho+\lambda d_{h_1}+\lambda f_{h_2})>0,\ (h_1=1,\ 2,\ \cdots,\ r;h_2=1,\ 2,\ \cdots,\ l).\end{cases}$

(b) In (2.1), setting $A=p$, $E=k$, $l=1$, $f_1=0$ and replacing $A+C$ by $A, B+D$ by $B, A+q$ by s together with appropriate changes in the parameters and then $v\to 0$ etc., we obtain the known result [(6), p. 585, (1)] :

$$(3.2)\ \int_0^1 x^{\rho-1}(1-x)^{\rho-1}{}_2F_1\left(\begin{matrix}\alpha,\ \beta\\ \frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\end{matrix}:x\right)G^{r,\ s}_{A,\ B}\left[ux^{\lambda}(1-x)^{\lambda}\Big|\begin{matrix}(a)\\(b)\end{matrix}\right]dx$$

$$=\frac{\pi 2^{1-2\rho}\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\right)}{\sqrt{\lambda}\,\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)}$$

$$\times G^{r,\ s+2\lambda}_{A+2\lambda,\ B+2\lambda}\left[\frac{u}{2^{2\lambda}}\left|\begin{matrix}\nabla(\lambda,\rho),\ \nabla\left(\lambda,\rho+\frac{1}{2}-\frac{1}{2}\alpha-\frac{1}{2}\beta\right),\ (a)\\ (b),\ \nabla\left(\lambda,\rho+\frac{1}{2}-\frac{1}{2}\alpha\right),\ \nabla\left(\lambda,\rho+\frac{1}{2}-\frac{1}{2}\beta\right)\end{matrix}\right.\right]$$

where λ is a positive integer, $\mathrm{Re}(\alpha+\beta)>-1$: $\nabla(m,n)=1-\frac{n}{m},\ 1-\frac{n+1}{m},\ \cdots\cdots,$

$1-\frac{n+m-1}{m}$ and

(i) $\begin{cases}2(r+s)>A+B,\ |\arg(u)|<\left[r+s-\frac{1}{2}(A+B)\right]\pi\\ \mathrm{Re}(\rho+\lambda b_j)>0,\ (j=1,\ 2,\ \cdots,\ r)\end{cases}$

(ii) $\begin{cases}A<B \text{ or else } A=B \text{ with } |u|<1,\\ \mathrm{Re}(\rho+\lambda b_j)>0,\ (j=1,\ 2,\ \cdots,\ r).\end{cases}$

(c) Substituting $p=A=m$, $B=n$, $q=k=C=E=l$, $r=l=1$, $D=F=p+1$, $d_1=f_1=0$ and replacing b_j, $1-c_j$, $1-d_j$, $1-e_j$ and $1-f_j$ by c_j, b_j, d_j, b_j' and d_j' respectively in (2.1), the s-function reduces to Kampé de Fériet's function (1.8) thus we obtain

$$(3.3)\quad \int_0^1 x^{\rho-1}(1-x)^{\rho-1}{}_2F_1\left(\begin{matrix}\alpha,\ \beta\\ \frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\end{matrix} : x\right) F^{m,\ l}_{n,\ p}\left[\begin{matrix}ux^\lambda(1-x)^\lambda\\ vx^\lambda(1-x)^\lambda\end{matrix}\right]dx$$

$$=\frac{\pi\, 2^{1-2\rho}\,\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\right)\prod_{i=0}^{\lambda-1}\Gamma\left(\frac{\rho+i}{\lambda}\right)\prod_{i=0}^{\lambda-1}\Gamma\left(\frac{\rho+\frac{1}{2}-\frac{1}{2}\alpha-\frac{1}{2}\beta+i}{\lambda}\right)}{\sqrt{\lambda}\,\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)\prod_{i=0}^{\lambda-1}\Gamma\left(\frac{\rho+\frac{1}{2}-\frac{1}{2}\alpha+i}{\lambda}\right)\prod_{i=0}^{\lambda-1}\Gamma\left(\frac{\rho+\frac{1}{2}-\frac{1}{2}\beta+i}{\lambda}\right)}$$

$$\times F^{m+2\lambda,\, l}_{n+2\lambda,\, p}\left[\begin{matrix}\Delta(\lambda,\rho),\ \Delta\left(\lambda,\rho+\frac{1}{2}-\frac{1}{2}\alpha-\frac{1}{2}\beta\right),\ a_m : b_l : b_l'\\ \Delta\left(\lambda,\rho+\frac{1}{2}-\frac{1}{2}\alpha\right),\ \Delta\left(\lambda,\rho+\frac{1}{2}-\frac{1}{2}\beta\right),\ c_n : d_p : d_p'\end{matrix}\left|\begin{matrix}\frac{u}{2^{2\lambda}}\\ \frac{v}{2^{2\lambda}}\end{matrix}\right.\right]$$

where λ is a positive integer, $\mathrm{Re}(\alpha+\beta)>-1$, $\mathrm{Re}(\rho)>0$ and $m+l<n+p+1$, $\{m+l=n+p+1$: then $|u|<1$, $|v|<1\}$ or if $m+l+1>n+p$, then $|\arg(v)|$, $|\arg(u)|<(m+l-n-p)\frac{\pi}{2}$.

4. The double-integral analogues.

Following the method of the preceding section we can easily obtain the double-integral analogues for generalized Meijer functions of two variables $S\left[\begin{matrix}ux^\lambda(1-x)^\lambda\\ vy^\mu(1-y)^\mu\end{matrix}\right]$ with Gauss's hypergeometric functions:

$$(4.1)\quad \int_0^1\int_0^1 x^{\rho-1}(1-x)^{\rho-1}y^{\sigma-1}(1-y)^{\sigma-1}{}_2F_1\left(\begin{matrix}\alpha,\beta\\ \frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\end{matrix} : x\right)$$

$$\times{}_2F_1\left(\begin{matrix}\gamma,\delta\\ \frac{1}{2}\gamma+\frac{1}{2}\delta+\frac{1}{2}\end{matrix} : y\right) S\left[\begin{matrix}ux^\lambda(1-x)^\lambda\\ vy^\mu(1-y)^\mu\end{matrix}\right]dxdy$$

$$=\frac{\pi^2\,\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\gamma+\frac{1}{2}\delta+\frac{1}{2}\right)2^{2-2\rho-2\sigma}}{\sqrt{(\lambda\mu)}\,\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\beta+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\gamma+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\delta+\frac{1}{2}\right)}$$

$$\times S\left[\begin{matrix}\left[\begin{matrix}p, & 0\\ A-p, & B\end{matrix}\right]\\ \left(\begin{matrix}q+2\lambda, & r\\ C-q, & D-r+2\lambda\end{matrix}\right)\\ \left(\begin{matrix}k+2\mu, & l\\ E-k, & F-l+2\mu\end{matrix}\right)\end{matrix}\left|\begin{matrix}(a):(b)\\ \Delta(\lambda,-\rho+1),\ \Delta\left(\lambda,-\rho+\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\right),\ (c):\\ (d),\ \Delta\left(\lambda,-\rho+\frac{1}{2}+\frac{1}{2}\alpha\right),\ \Delta\left(\lambda,-\rho+\frac{1}{2}+\frac{1}{2}\beta\right)\\ \Delta(\mu,-\sigma+1),\ \Delta\left(\mu,-\sigma+\frac{1}{2}\gamma+\frac{1}{2}\delta+\frac{1}{2}\right),\ (e):\\ (f),\ \Delta\left(\mu,-\sigma+\frac{1}{2}+\frac{1}{2}\gamma\right),\ \Delta\left(\mu,-\sigma+\frac{1}{2}+\frac{1}{2}\delta\right)\end{matrix}\right|\begin{matrix}\frac{u}{2^{2\lambda}}\\ \frac{v}{2^{2\mu}}\end{matrix}\right]$$

where λ, μ are positive integers, $\mathrm{Re}(\alpha+\beta)>-1$, $\mathrm{Re}(\gamma+\delta)>-1$ and

$$\text{(i)}\begin{cases} 2(p+q+r)>A+B+C+D, & |\arg(u)|<\left[p+q+r-\frac{1}{2}(A+B+C+D)\right]\pi, \\ 2(p+k+l)>A+B+E+F, & |\arg(v)|<\left[p+k+l-\frac{1}{2}(A+B+E+F)\right]\pi \\ \mathrm{Re}(\rho+\lambda d_{h_1})>0, & (h_1=1,\ 2,\ \cdots,\ r) \\ \mathrm{Re}(\sigma+\mu f_{h_2})>0, & (h_2=1,\ 2,\ \cdots,\ l) \end{cases}$$

or

$$\text{(ii)}\begin{cases} A+C<B+D,\ A+E<B+F, \\ \text{or else } A+C=B+D,\ A+E=B+F \text{ with } |u|<1,\ |v|<1, \\ \mathrm{Re}(\rho+\lambda d_{h_1})>0,\ (h_1=1,\ 2,\ \cdots,\ r) \\ \mathrm{Re}(\sigma+\mu f_{h_2})>0,\ (h_2=1,\ 2,\ \cdots,\ l). \end{cases}$$

(4.2)
$$\int_0^1\int_0^1 x^{\rho-1}(1-x)^{\rho-\beta}y^{\sigma-1}(1-y)^{\sigma-\delta}\ {}_2F_1\left(\begin{matrix}\alpha, 1-\alpha \\ \beta\end{matrix};x\right)$$
$$\times {}_2F_1\left(\begin{matrix}\gamma, 1-\gamma \\ \delta\end{matrix};y\right) S\left[\begin{matrix}ux^{\lambda}(1-x)^{\lambda} \\ vy^{\mu}(1-y)^{\mu}\end{matrix}\right]dxdy$$
$$=\frac{\pi^2\Gamma(\beta)\Gamma(\delta)2^{2-2\rho-2\sigma}}{\sqrt{(\lambda\mu)}\ \Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\alpha+\frac{1}{2}\beta\right)\Gamma\left(\frac{1}{2}\gamma+\frac{1}{2}\delta\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\gamma+\frac{1}{2}\delta\right)}$$
$$\times S\left[\begin{matrix}\begin{bmatrix}p, & 0 \\ A-p, & B\end{bmatrix} \\ \begin{pmatrix}q+2\lambda, & r \\ C-q, & D-r+2\lambda\end{pmatrix} \\ \begin{pmatrix}k+2\mu, & l \\ E-k, & F-l+2\mu\end{pmatrix}\end{matrix}\left|\begin{matrix}(a):(b) \\ \Delta(\lambda,-\rho+1),\ \Delta(\lambda,-\rho+\beta),\ (c): \\ (d),\ \Delta\left(\lambda,\ -\rho-\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\right),\ \Delta\left(\lambda,-\rho+\frac{1}{2}\alpha+\frac{1}{2}\beta\right) \\ \Delta(\mu,-\sigma+1),\ \Delta(\mu,-\sigma+\delta),\ (e): \\ (f),\ \Delta\left(\mu,-\sigma-\frac{1}{2}\gamma+\frac{1}{2}\delta+\frac{1}{2}\right),\ \Delta\left(\mu,-\sigma+\frac{1}{2}\gamma+\frac{1}{2}\delta\right)\end{matrix}\right|\begin{matrix}\frac{u}{2^{2\lambda}} \\ \frac{v}{2^{2\lambda}}\end{matrix}\right]$$

valid for λ, μ positive integers, $\mathrm{Re}(\beta)>0$, $\mathrm{Re}(\delta)>0$ and

$$\text{(i)}\begin{cases} 2(p+q+r)>[A+B+C+D], & |\arg(u)|<\left[p+q+r-\frac{1}{2}(A+B+C+D)\right]\pi, \\ 2(p+k+l)>[A+B+E+F], & |\arg(v)|<\left[p+k+l-\frac{1}{2}(A+B+E+F)\right]\pi \\ \mathrm{Re}(\rho+\lambda d_{h_1})>0, & \mathrm{Re}(\rho-\beta+\lambda d_{h_1})>-1,\ (h_1=1,\ 2,\cdots,\ r) \\ \mathrm{Re}(\sigma+\mu f_{h_2})>0, & \mathrm{Re}(\sigma-\delta+\mu f_{h_2})>-1,\ (h_2=1,\ 2,\ \cdots,\ l) \end{cases}$$

or

$$\text{(ii)}\begin{cases} A+C<B+D,\ A+E<B+F, \\ \text{or else } A+C=B+D,\ A+E=B+F \text{ with } |u|<1,\ |v|<1, \\ \mathrm{Re}(\rho+\lambda d_{h_1})>0,\ \mathrm{Re}(\rho-\beta+\lambda d_{h_1})>-1,\ (h_1=1,\ 2,\ \cdots,\ r), \\ \mathrm{Re}(\sigma+\mu f_{h_2})>0,\ \mathrm{Re}(\sigma-\delta+\mu f_{h_2})>-1,\ (h_2=1,\ 2,\ \cdots,\ l). \end{cases}$$

These results provide us with double-integral analogues of our earlier results

(2.1) and (2.2) respectively.

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