HAUSDORFF METRIC ON THE FAMILY OF WEAKLY COMPACT SETS IN A BANACH SPACE

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In this note we prove that the family of non empty weakly compact convex subsets of a weakly complete Banach space is a complete metric space if the Hausdorff metric is given. It is also shown that the collection of strictly convex and smooth convex bodies form a dense subset of the metric space.

1. Hausdorff metric.

Let X be a weakly complete Banach space and let $\mathscr{K}$ be the collection of non empty weakly compact convex subsets of X. For $A, B \in \mathscr{K}$, define

$$d(A, B) = \inf \{r: A \subset V_r(B) \text{ and } B \subset V_r(A)\}$$

where

$$V_r(A) = \{x \in X: \inf \{\|x-a\|: a \in A\} \leq r\}.$$

Since every weakly compact convex subset of a Banach space is bounded and closed in norm [1, Theorem V.6.1 and Theorem V.3.13] clearly the real valued function $d: \mathscr{K} \times \mathscr{K} \to \mathscr{R}$ is a metric on $\mathscr{K}$ (also see [2, p. 131]).

LEMMA 1. *If $\{A_n\}$ is a Cauchy sequence in the metric space $\langle \mathscr{K}, d \rangle$, then the union $\bigcup_{n=1}^{\infty} A_n$ is weakly sequentially compact.*

PROOF. Suppose that $\{a_n\}$ is a sequence in $\bigcup_{n=1}^{\infty} A_n$. If $\{a_n\}$ is a subset of finitely many A_n 's, then clearly $\{a_n\}$ has a subsequence which converges weakly to an element of A_n for some n [1, Theorem V.6.1]. Suppose that no finite union of $\{A_n\}$ contains the sequence $\{a_n\}$. Without loss of generality we may assume that $a_n \in A_n - \bigcup_{j<n} A_j$. Given $\varepsilon > 0$, choose N such that $d(A_i, A_j) < \frac{\varepsilon}{5}$ if $i, j \geqq N$.
For each $j \geqq N$ there is a $b_j \in A_N$ such that $\|a_j - b_j\| < \frac{\varepsilon}{4}$ by the definition of the metric d. Since A_n is weakly compact the sequence $\{b_j\}$ has a weakly convergent subsequence, say $\{b_j'\} \to b \in A_N$ weakly. Let $\{a_j'\}$ be a subsequence of $\{a_n\}$ corresponding to $\{b_j'\}$. Choose $N_1 \geqq N$ for all $f \in X^*$ such that $|f(b_j') - f(b)| < \frac{\varepsilon}{4}$ whe-n ever $j > N_1$. If $m, n > N_1$, then for the functional f

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$$|f(a_m')-f(a_n')|\leqq|f(a_m')-f(b_m')|+|f(b_m')-f(b)|$$
$$+|f(b)-f(b_n')|+|f(b_n')-f(a_n')|<\varepsilon.$$

Therefore $\{a_j'\}$ is a weak Cauchy sequence and it converges.

LEMMA 2. *If* $\{D_n\}$ *is a Cauchy sequence in* $\mathscr{K}$ *with* $D_n\supset D_{n+1}$ *then* $D=\bigcap_{n=1}^{\infty} D_n$ *is a member of* $\mathscr{K}$ *and* $\lim_{n\to\infty} D_n=D$.

PROOF. Clearly $D\neq\phi$ (finite intersection property) and closed, hence it is weakly compact. Suppose that $\lim_{n\to\infty} D_n\neq D$, then there is an $\varepsilon>0$ and a subsequence $\{D_n'\}$ of $\{D_n\}$ such that

$$d(D_n',\ D)=\inf\ \{r\colon D_n'\subset V_r(D)\}\geqq\varepsilon.$$

Choose a number N such that

$$d(D_N,\ D)\geqq\varepsilon \text{ and } d(D_i,\ D_j)<\frac{\varepsilon}{2},\ i,\ j\geqq N.$$

From $d(D_N,\ D)\geqq\varepsilon$, we may choose $x_N\in D_N$ such that

$$\inf\ \{\|x_N-x\|\colon\ x\in D\}\geqq\frac{2}{3}\varepsilon.$$

Since $\{D_n\}$ is decreasing and $d(D_N,\ D_{N+j})<\frac{\varepsilon}{2}$, all j, there is a sequence $\{x_j\}$ such that $x_j\in D_{N+j}$ and $\|x_N-x_j\|<\frac{\varepsilon}{2}$. Let x_0 be a weak limit point of a subsequence of $\{x_j\}$ then $x_0\in D_N$ and $\|x_N-x_0\|\leqq\frac{\varepsilon}{2}$ (This follows from the fact that $\{x_j\}$ is a sequence of weakly compact convex set $\{x\colon\ x\in D_N,\ \|x-x_N\|\leqq\frac{\varepsilon}{2}\}$), therefore $x_0\notin D$. Now we will show that $x_0\in D$ and yield a contradiction. Since the sequence $\{x_j,\ x_{j+1},\ x_{j+2},\cdots\cdots\}$ is a subset of D_{N+j} for each $j=1,\ 2,\ 3,\ \cdots\cdots$, $x_0\in D_{N+j}$ by the weak compactness and $x_0\in\bigcap_{j=1}^{\infty}D_{N+j}=\bigcap_{n=1}^{\infty}D_n=D$.

THEOREM 1. *If* X *is a weakly complete Banach space, then the family* $\mathscr{K}$ *of all non empty weakly compact convex subsets of* X *is a complete metric space in the Hausdorff metric.*

PROOF. Suppose that $\{A_n\}$ is a Cauchy sequence in $\mathscr{K}$ and let $D_n=\overline{\text{co}}\ (\bigcup_{i=n}^{\infty}A_i)$ where $\overline{\text{co}}(S)$ denotes the closed convex hull of a set S. Since the closed convex hull of a weakly compact set is weakly compact [1, Theorem V.6.4] and a convex set in a Banach space is weakly closed if and only if it is closed in norm topology [1, Theorem V.3.13], $D_n=\overline{\text{co}}\ (\bigcup_{i=n}^{\infty} A_i)$ is weakly compact by Lemma 1. We claim that $\{D_n\}$ is a decreasing Cauchy sequence and by Lemma 2 $\lim_{n\to\infty} D_n=D$.

Indeed, given $\varepsilon>0$ let N be a number with $d(A_i, A_j)<\frac{\varepsilon}{3}, i, j\geqq N$. Suppose that $k\geqq N$ and $x\in \text{co} \bigcup_{i=k}^{\infty} A_i$, then $x=\sum_{j=1}^{N(k)} t_j x_j$ where $x_j\in\bigcup_{i=k}^{\infty} A_i$, $\sum t_j=1$, $0<t_j<1$. Let $a_j\in A_k$ and $\|a_j-x_j\|<\frac{\varepsilon}{3}$. Since $\sum_{j=1}^{N(k)} t_j a_j\in A_k$ and $\|x-\sum_j t_j a_j\|<\frac{\varepsilon}{3}$, $D_k\subset V_{\frac{\varepsilon}{3}}(A_k)$ and $d(D_k, A_k)\leq\frac{\varepsilon}{3}$. If $i, j\geqq N$,

$$d(D_i, D_j)\leqq d(D_i, A_i)+d(A_i, A_j)+d(A_j, D_j)<\varepsilon$$

In the above proof we have shown that given $\varepsilon>0$, there is a number N such that $d(A_k, D_k)<\varepsilon$ whenever $k\geqq N$. Hence $\lim_{n\to\infty} A_n=\lim_{n\to\infty} D_n=\bigcap_{n=1}^{\infty} D_n$.

2. An approximation theorem.

Approximation of convex sets of separable Banach spaces was studied in Klee [3]. Here we prove that a convex body in a reflexive space can be approximated by a strictly convex and smooth one. Let C be a convex body (a bounded closed convex subset with non empty interior) of a Banach space. A functional $f_0\in X^*$ is called a supporting functional of C at a point $x_0\in C$ if $f_0(x_0)=r=\sup\{\text{Re } f_0(x): x\in C\}$ and the hyperplane $\{x: x\in X \text{ and } f_0(x)=r\}$ is said to be supporting C at x_0. If every boundary point x_0 of a convex body C satisfies the property that each supporting hyperplane of C at x_0 intersects C at exactly one point x_0, then C is said to be strictly convex. A convex body C is called smooth if there is a unique hyperplane supporting C at each boundary point of C. The norm of a Banach space is called *strictly convex* (smooth, respectively) if every boundary point of the unit ball of the space is strictly convex (smooth, respectively) [3], [4].

THEOREM 2. *Let C be a convex body of a (non separable) reflexive Banach space X and let p be an interior point of C. Given $0<\varepsilon<1$ there is a strictly convex and smooth convex body D such that $p+\varepsilon(C-p)\subset D\subset C$ where $c+p$ $=\{c+p: c\in C\}$.*

PROOF. We may assume that $p=0$. Since X is reflexive there is an equivalent strictly convex and smooth norm $\|.\|$ on X [4, Theorem 5.2] and let U be the closed unit ball of X under the norm $\|.\|$. Hence U is a strictly convex and smooth convex body of X. Let $r>0$ such that $rU=\{rx: x\in U\}\subset\frac{1-\varepsilon}{2} C$ and let $D_0=rU+\frac{1+\varepsilon}{2} C$. Clearly D_0 is a convex set with non empty interior. Hence, to see the set D_0 is closed it is enough to show that it is weakly closed. Suppose that $\{d_n\}$ is a seqence in D_0 with $d_n=u_n+c_n$ where $u_n\in rU, c_n\in\frac{1+\varepsilon}{2}C$. Since both U and C are weakly compact, there are subsequences $\{u_n'\}$ and $\{c_n'\}$ of $\{u_n\}$ and $\{c_n\}$, respectively, and $u_n'\to u\in rU$ and $c_n'\to c\in\frac{1+\varepsilon}{2}C$ weakly, respectively. Hence, the corresponding subsequence $d_n'\to u+c$ weakly and $u+c\in D_0$, so D_0 is weakly compact. Smoothness of D_0 follows immediately from the smoothness of rU. Now by the proof of Theorem 1.5 and Lemma 1.4 of Klee [3] there is a homeomorphism T on X such that $D=TD_0$ is strictly convex, smooth and $\varepsilon C\subset D\subset C$.

COROLLARY. *Let $\mathscr{K}$ be the family of non empty weakly compact convex subsets of a reflexive Banach space X with the Hausdorff metric. Then the collection of strictly convex and smooth convex bodies of X is dense in $\mathscr{K}$.*

PROOF. Given $\varepsilon>0$ and $K\in\mathscr{K}$ it is clear that $V_\varepsilon(K)$ is a convex body. Let $p\in K$. Then there exists a strictly convex and smooth convex body D such that $p+\varepsilon(V_\varepsilon(K)-p)\subset D\subset V_\varepsilon(K)$. Now clearly $d(K, D)<\varepsilon$.

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