AN CERTAIN HYPERSURFACES WITH AN (f,g,u,v,λ)-STRUCTURE IN A SASAKIAN MANIFOLD

By Un Kyu Kim

§0. Introduction

The (f,g,u,v,λ)-structures have been studied by many authors [2,5,6]. It is well known that the hypersurface of an almost contact manifold admits an (f,g,u,v,λ)-structure [2,6]. D.E. Blair, G.D. Ludden and K. Yano have studied the hypersurface of an odd-dimensional sphere in the case of $f\circ h=h\circ f$ and $f\circ h=-h\circ f$, where h is the second fundamental tensor of the hypersurface [1].

On the other hand, S. Yamaguchi [4] have studied the hypersurface of a Sasakian manifold and obtained a result in the case of $f\circ h=h\circ f$

In this present paper, we investigate the hypersurfaces of certain Sasakian manifold in the case of $f\circ h=-h\circ f$.

§1. Hypersurfaces in a Sasakian manifold

Let $\tilde{M}$ be $(2n+1)$-dimensional almost contact metric manifold covered by a system of coordinate neighborhoods $\{\tilde{U}, X^\kappa\}$, where here and in the sequel indices κ, μ, ν, $\lambda\cdots$ run over the range $\{1, 2, \cdots, 2n+1\}$ and let $(F_\kappa^{\lambda}, G_{\mu\lambda}, V_\lambda)$ be the almost metric structure, that is,

(1.1) $F_\kappa{}^\mu F_\mu{}^\lambda=-\delta_\kappa{}^\lambda+V_\kappa V^\lambda$,

(1.2) $V_\kappa F_\lambda{}^\kappa=0$, $F_\lambda{}^\kappa V^\lambda=0$,

(1.3) $V^\lambda V_\lambda=1$ and

(1.4) $G_{\gamma\beta}F_\lambda{}^\kappa F_\mu{}^\beta=G_{\lambda\mu}-V_\lambda V_\mu$

If $\tilde{M}$ is a Sasakian manifold, then

(1.5) $\nabla_\mu V^\kappa=F_\mu{}^\kappa$, $\nabla_\mu F_\lambda{}^\kappa=-G_{\mu\lambda}V^\kappa+\delta_\mu{}^\kappa V_\lambda$

and for the curvature tensor $\tilde{R}_{\nu\mu\lambda\kappa}$ of $\tilde{M}$ we have

(1.6) $V^\kappa\tilde{R}_{\nu\mu\lambda\kappa}=V_\nu G_{\mu\lambda}-V_\mu G_{\nu\lambda}$

, where we denote by $\left\{{\kappa \atop \mu\lambda}\right\}$ and ∇_μ the christoffel symbols formed with the Riemannian metric of $G_{\mu\lambda}$ of $\tilde{M}$ and the operator of covariant differentiation with respect to $\left\{ {\kappa \atop \mu\lambda} \right\}$ respectively.

Let M be a $2n$-dimensional differentiable manifold which is covered by a system of coordinate neighborhoods $\{U, x^h\}$, where here and in the sequel the indicies $h, i, j, k, \cdots$, run over the range $\{1, 2, \cdots, 2n\}$ and differentiably imbedded in $\tilde{M}$ as a hypersurface by the equations $X^\kappa = X^\kappa(X^h)$

We put

$$B_i^\kappa = \partial_i X^\kappa, \quad \partial_i = \frac{\partial}{\partial x^i},$$

and choose a unit C^κ of $\tilde{M}$ normal to M in such a way that $2n+1$ vectors B_i^κ and C^κ give the positive orientation of $\tilde{M}$ as the following

(1.7) $F_\lambda^{\ \kappa} B_i^{\ \lambda} = f_i^{\ h} B_h^{\ \kappa} + u_i C^\kappa$,

(1.8) $F_\lambda^{\ \kappa} C^\lambda = -u^i B_i^{\ \kappa}$

(1.9) $V^\kappa = B_i^{\ \kappa} v^i + \lambda C^\kappa$

where $f_i^{\ h}$ is a tensor field of type (1,1), u_i and v_i are 1-forms of M, λ is a function on M and $u^i = u_j g^{ji}$, $v_i = v_j g^{ji}$ for the induced Riemannian metric g_{ji} on M from that of $\tilde{M}$.

Then the hypersurface M admits an (f, g, u, v, λ)-structure [6], that is,

$$f_i^{\ j} f_j^{\ h} = -\delta_i^h + u_i u^h + v_i v^h$$

$$f_i^{\ h} u^i = -\lambda v^h, \quad f_i^{\ h} v^i = \lambda u^h,$$

$$f_h^{\ i} u_i = \lambda v_h, \quad f_h^{\ i} v_i = -\lambda u_h$$

$$u^i u_i = 1 - \lambda^2, \quad v^i v_i = 1 - \lambda^2$$

$$u^i v_i = 0, \quad v^i u_i = 0$$

$$g_{ij} f_h^{\ i} f_l^{\ j} = g_{hl} - u_h u_l - v_h v_l$$

We denote by $\left\{ {h \atop ij} \right\}$ and ∇_i the christoffel symbols formed with g_{ji} and the operator of covariant differentiation withrespect to $\left\{ {h \atop ji} \right\}$ respectively. Then the equations of Gauss and those of Weingarten are

(1.11) $\nabla_j B_i^{\ \kappa} = h_{ji} C^\kappa$ and $\nabla_j C^\kappa = -h_j^{\ i} B_i^{\ \kappa}$ respectively, where h_{ji} is the second fundamental tensor and $h_j^{\ i} = h_{jl} g^{li}$

We have from (1.5) and (1.7)~(1.11) the following

(1.12) $\nabla_j f_i^h = -h_{ji}u^h + h_j^h u_i - g_{ji}v^h + \delta_j^h v_i,$

(1.13) $\nabla_j u_i = -h_{jl}f_i^l - \lambda g_{ji},$

(1.14) $\nabla_j v^h = f_j^h + \lambda h_j^h,$

(1.15) $\nabla_j \lambda = u_j - h_{ji}v^i.$

Now we assume that the hypersurface M of the Sasakian manifold $\tilde{M}$ satisfies $f_i^h h_h^j = -h_i^h f_h^j$ and $\lambda(1-\lambda^2)$ is almost everywhere non-zero.

The condition $f_i^h h_h^j = -h_i^h f_h^j$ is equivalent to

(1.16) $f_j^k h_{ki} = f_i^k h_{kj}.$

If we transvect (1.16) with $v^i u^j$, then we have

(1.17) $h_{ij}u^i u^j = -h_{ij}v^i v^j.$

Transvecting (1.16)with f_k^j and taking account of (1.10), we have

$$(u^k u^l + v^k v^l)h_{li} = (u_i u^l + v_i v^l)h_l^k,$$

from which

$$(1-\lambda^2)h_{ij}u^j = h_{lj}u^l u^j u_i + h_{lj}u^l v^j v_i,$$

$$(1-\lambda^2)h_{ij}v^j = h_{lj}u^l v^j u_i + h_{lj}v^l v^j v_i.$$

From (1.17) these equations can be written as

(1.18) $h_{ij}u^j = \beta u_i - \alpha v_i,$

$$h_{ij}v^j = \beta u_i - \alpha v_i,$$

where α and β are defined by

$$h_{ji}u^j u^i = \alpha(1-\lambda^2),$$

$$h_{ji}u^j v^i = \beta(1-\lambda^2).$$

Moreover, from (1.16) we can easily show that $h_i^i = 0$, that is, the hypersurface M is minimal.

Differentiating the second equation of (1.18) covariantly along M, we obtain

(1.19) $(\nabla_k h_{ij})v^j + h_{ij}[f_k^j + \lambda h_j^k] = (\nabla_k \beta)u_i + \beta(-h_k^l f_{li} - \lambda g_{ki}) - (\nabla_k \alpha)v_i - \alpha(f_{ki} + \lambda h_{ki}).$

On the other hand, operating ∇_k to (1.15) and taking account of (1.13) and (1.14), we have

(1.20) $\nabla_k\nabla_j\lambda=-\lambda g_{kj}-f_j^l h_{lk}-(f_k^h+\lambda h_k^h)h_{hj}-v^h\nabla_k h_{jh}.$

If we subtract (1.20) from the equation obtained by interchanging the indicies k and i in (1.19), we obtain

(1.22)) $(\nabla_k\beta)u_i-(\nabla_i\beta)u_k-[(\nabla_k\alpha)v_i-(\nabla_i\alpha)v_k]-2\alpha f_{ki}=0,$

because of (1.21) and (1.16).

Transvecting (1.22) with u^i and making use of (1.10) we get

(1.23) $\nabla_k\beta=\frac{1}{1-\lambda^2}\{u^i(\nabla_i\beta)u_k+(2\lambda\alpha-u^i\nabla_i\alpha)v_k\}.$

Substituting (1.23) into(1.22), we find

(1.24) $$\frac{1}{1-\lambda^2}\{2\lambda\alpha v_k u_i-2\lambda\alpha v_i u_k-(\nabla_j\alpha)u^j u_i v_k+u^j(\nabla_j\alpha)v_i u_k\}$$
$$-\{(\nabla_k\alpha)v_i-(\nabla_i\alpha)v_k\}-2\alpha f_{ki}=0$$

If we transvect (1.24) with v^i we have

(1.25) $\nabla_k\alpha=\frac{1}{1-\lambda^2}\{u^j(\nabla_j\alpha)u_k+v^j(\nabla_j\alpha)v_k\}.$

Substituting (1.25) into (1.24), we obtain

$$\frac{1}{1-\lambda^2}\{2\lambda\alpha v_k u_i-2\lambda\alpha v_i u_k-(\nabla_j\alpha)u^j v_i u_k+(\nabla_j\alpha)u^j v_i u_k\}$$
$$-\frac{1}{1-\lambda^2}\Big[\{(\nabla_j\alpha)u^j u_k+(\nabla_j\alpha)v^j v_k\}v_i-\{(\nabla_j\alpha)u^j u_i+(\nabla_j\alpha)v^j v_i\}v_k\Big]-2\lambda f_{ki}=0,$$

from which

(1.26) $\lambda\alpha(v_k u_i-v_i u_k)=\alpha(1-\lambda^2)f_{ki}.$

Transvecting (1.26) with f^{ij} and using (1.10), we find

$$\alpha(1-\lambda^2)[-\delta_k^j+u_k u^j+v_k v^j]=-\lambda^2\alpha[v_k v^j+u_k u^j],$$

from which, we have $\alpha=0$ if $n>1$. Thus, the equations of (1.18) can be written as

(1.27) $$h_{ij}u^j=\beta v_i,\qquad h_{ij}v^j=\beta u_i.$$

§2. Certain hypersurface of a Sasakian manifold

In this section we assume that the tangent space of the hypersurface M is invariant under the curvature transformation of a Sasakian manifold $\tilde{M}$. The equation of Codazzi are $\nabla_k h_{ji}-\nabla_j h_{ki}=0$.

Gauss equation is given by

(2.1) $\tilde{R}_{\nu\mu\lambda\kappa} B_\nu^{\ \kappa} B_j^{\ \mu} B_i^{\ \lambda} B_h^{\ \kappa} = R_{kjih} - (h_{kh}h_{ji} - h_{jk}h_{ki})$.

Transvecting (2.1) with v^j, we have

$$\tilde{R}_{\nu\mu\lambda\kappa} B_k^{\ \nu} B_j^{\ \mu} B_i^{\ \lambda} B_h^{\ \kappa} v^j = \tilde{R}_{\nu\mu\lambda\kappa} B_k^{\ \kappa}(V^\mu - \lambda\, C^\mu) B_i^{\ \lambda} B_h^{\ \kappa}$$

$$= \tilde{R}_{\nu\mu\lambda\kappa} V^\mu B_k^{\ \nu} B_i^{\ \lambda} B_h^{\ \kappa} = (V_\lambda G_{\kappa\nu} - V_\kappa G_{\lambda\nu}) B_k^{\ \nu} B_i^{\ \lambda} B_h^{\ \kappa}$$

$$= v_i g_{kh} - v_h g_{ki} = R_{kjih} v^j - \beta(h_{kh}u_i - h_{ki}u_h)$$

because of (1.6), (1.9) and $\tilde{R}_{\nu\mu\lambda\kappa} B_k^{\ \nu} B_j^{\ \mu} B_i^{\ \lambda} C^\kappa = 0$,

Thus, we find

(2.2) $R_{kjih} v^j = v_i g_{kh} - v_h g_{ki} + \beta(h_{kh}u_i - h_{ki}u_h)$.

Differentiating the first equation of (1.27) covariantly, we find

(2.3) $(\nabla_k h_{ij})u^j + h_{ij}(\nabla_k u^j) = (\nabla_k \beta)v_i + \beta(f_{ki} + \lambda h_{ki})$.

If we subtract (2.3) from the equation obtained by interchanging the indices k and i in (2.3), we obtain

(2.4) $-2f_{kj}h_{il}h^{jl} = (\nabla_k \beta)v_i - (\nabla_i \beta)v_k + 2\beta f_{ki}$

Transvecting (2.4) with v^i we have

$\nabla_k \beta = \frac{1}{1-\lambda^2}\{(2\beta\lambda + 2\beta^2\lambda)u_k + (\nabla_i \beta)v^i u_k\}$, or

(2.5) $\nabla_k \beta = \frac{1}{1-\lambda^2}(2\beta\lambda + 2\beta^2\lambda)u_k$ by virtue of (1.23).

Substituting (2.5) into (2.4), we find

$-f_k^{\ j} h_{il} h_j^{\ l} = \frac{1}{1-\lambda^2}\lambda(\beta + \beta^2)(u_k v_i - u_i v_k) + \beta f_{ki}$.

Transvecting the above equation with $f_l^{\ k}$, we obtain

$-(-\delta_l^{\ j} + u_l u^j + v_l v^j) h_{ih} h_j^{\ h} = \frac{1}{1-\lambda^2}(\beta + \beta^2)\lambda^2 [v_l v_i + u_l u_i] + \beta[-g_{li} + u_l u_i + v_l v_i]$,

from which

(2.6) $(1-\lambda^2)(h_l^{\ h} h_{ih} + \beta g_{li}) = \beta(\beta+1)(u_j u_i + v_j v_i)$.

We here assume that the sectional curvature $\kappa(x)$ with respect to the section spanned by u^h and v^h is constant at every point x of M. Then the sectional curvature $\kappa(x)$ with respect to the section spanned by u^h and v^h is given by

$$\kappa(x) = \frac{R_{kjih}\, u^k v^j u^i v^h}{-v_j v^j u_i u^i} = 1 - \beta^2,$$

because of (2.2)

Thus, $\beta=0$ or $\beta=-1$ by virtue of (2.5).

Case 1. $\beta=0$.

From (2.6)we obtain $h_{ij}=0$. Hence we have

$\nabla_j\nabla_i\lambda=-\lambda g_{ji}$ from (1.13) and (1.15). By Obata's theorem [3], if M is complete orientable hypersurface in $\tilde{M}$, then M is isometric with a sphere $S^{2n}(1)$

Care 2. $\beta=-1$.

In this case we have from (2.6)

(2.7) $h_i^l h_l^j=\delta_i^j$

$h_i^i=0$, (2.7) and $\nabla_k h_{ij}=0$ show that M^{2n} is a product $N\times N'$ of N and N' both of the same dimension n. Thus, we cover N by a system of coordinate neighborhoods $\{V : y^a\}$ and N' by a system of coordinate neighborhoods $\{W : y^r\}$ and consequently $N\times N'$ by $\{V\times W ; y^h\}$. Then the metrvc tensor g has components of the form

(2.8) $g_{ji}=\begin{pmatrix} g_{cb}(y^a) & 0 \\ 0 & g_{ts}(y^r) \end{pmatrix}$

and h_i^h those of the form

(2.9) $h_j^i=\begin{pmatrix} \delta_a^{\ b} & 0 \\ 0 & -\delta_r^{\ s} \end{pmatrix}$

Thus, from (1.16), we see that f_i^h has components of the form

(2.10) $f_i^h=\begin{pmatrix} 0 & f_a^{\ s} \\ f_r^{\ b} & 0 \end{pmatrix}$

Since $\nabla_j\lambda=(1-\beta)u_j=2u_j$, we have

$$\nabla_j\nabla_i\lambda=-h_{jk}f_i^k-2\lambda g_{ji}.$$

Consequently, we have

(2.11) $\nabla_c\nabla_b\lambda=-2\lambda g_{cb}$

and

(2.12) $\nabla_t\nabla_s\lambda=-2\lambda g_{ts}$

because of (2.9) and (2.10).

$a, b, c, \cdots\cdots$ run over $1, 2, \cdots\cdots, n$

$r, s, t, \cdots\cdots$ run over $n+1, n+2, \cdots\cdots, 2n$

The submanifold N and N' being both complete, the theorem of Obata [3], (2.11) and (2.12) show that N and N′ are isometric to $S^n\left(\frac{1}{\sqrt{2}}\right)$. Thus M^{2n} is isometric to $S^n\left(\frac{1}{\sqrt{2}}\right)\times S^n\left(\frac{1}{\sqrt{2}}\right)$. Hence we have the following

THEOREM *Let M be a complete orientable hypersurface of a Sasakian manifold* $\tilde{M}$ *which satisfies* $f\circ h=-h\circ f$, $\lambda(1-\lambda^2)$ *being almost everywhere non-zero and the tangent space of M being invariant under the curvature transformation of* $\tilde{M}$. *If the sectional curvature* $\kappa(x)$ *with respeat to the section spanned by* u^h *and* v^h *is constant at every point* x *of M, then M is isometric with a sphere* $S^{2n}(1)$ *or M a product of* $S^n\left(\frac{1}{\sqrt{2}}\right)$ *and* $S^n\left(\frac{1}{\sqrt{2}}\right)$

BIBLIOGRAPHY

[1] Blair. D.E. G.D. Ludden and K. Yano, *Hypersurfaces of odd-dimensional sphere* to appear in J. of Diff. Geom.

[2] Blair. D.E. G.D. Ludden and K.Yano, *Induced structures on submanifolds,* Kōdai Math. Sem. Rep. 22 (1970) 188—198

[3] Obata, M., *Certain conditions for a Riemannian manifold to be isometric with a sphere,* J. Math Soc Japan 14 (1962) 333—340

[4] Yamaguchi. S., *On hypersurfaces in Sasakian manifolds,* Kōdai Math. Sem. Rep. 21 (1969) 64—72

[5] Yano. K and U-Hang Ki, *On quasi-normal* (f, g, u, v, λ) *structure,* to appear in Kōdai Math. Sem. Rep.

[6] Yano. K and M. Okumura, *On* (f, g, u, v, λ)*-Structure.* Kōdai Math. Sem. Rep. 22 (1970). 401—423

[7] Yano. K and M. Okumura, *On normal* (f, g, u, v, λ)*-Structures on submaniifolds of codimension* 2 *in an even dimensional Euclidean space,* Kōdai Math. Sem. Rep. 23(1971) 172—197.