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STRONGLY R_1 SPACES

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1. Introduction

In [8] a separation axiom between Hausdorff and Urysohn, called strongly Hausdorff, was introduced and used to investigate the cardinality of discrete subsets of Hausdorff spaces.

DEFINITION 1.1. A Hausdorff space (X, T) is *strongly Hausdorff* iff for each infinite subset A of X, there exists a sequence $\{U_n\}_{n\in N}$ of disjoint open sets such that $A\cap U_n\neq\phi$ for all $n\in N$.

In this paper strongly Hausdorff is generalized to strongly R_1, properties of strongly R_1 spaces are investigated, and T_0-identification spaces are further investigated and used to extend known results for strongly Hausdorff spaces to strongly R_1 spaces.

Throughout this paper N is used to denote the set of natural numbers.

2. R_1, strongly R_1, and T_0-identification spaces

LEMMA 2.1. *A Hausdorff space (X, T) is strongly Hausdorff iff for each sequence $\{x_n\}_{n\in N}$ such that $x_n=x_m$ iff $n=m$, there exists a sequence $\{U_n\}_{n\in N}$ of disjoint open sets such that $U_m\cap\{x_n|n\in N\}\neq\phi$ for all $m\in N$.*

The straightforward proof is omitted.

DEFINITION 2.1. A space is (X, T) is R_1 iff for $x, y\in X$ such that $\overline{\{x\}}\neq\overline{\{y\}}$, there exist disjoint open sets U and V such that $\overline{\{x\}}\subset U$ and $\overline{\{y\}}\subset V$ [5].

DEFINITION 2.2. A R_1 space (X, T) is *strongly* R_1 iff for each sequence $\{x_n\}_{n\in N}$ such that $\overline{\{x_n\}}=\overline{\{x_m\}}$ iff $n=m$, there exists a sequence $\{U_n\}_{n\in N}$ of disjoint open sets such that $U_m\cap\{x_n|n\in N\}\neq\phi$ for all $m\in N$.

In [5] it was shown that a space is Hausdorff iff it is R_1 and T_0. This result can be used to obtain the following result.

THEOREM 2.1. *A space is strongly Hausdorff iff it is strongly R_1 and T_0.*

The example in [10], which is Hausdorff but not strongly Hausdorff, shows that R_1 is weaker than strongly R_1.

DEFINITION 2.3. Let (X, T) be a space and let R be the equivalence relation on X defined by xRy iff $\overline{\{x\}}=\overline{\{y\}}$. Then the *$T_0$-identification space* of $(X, T)$is $(X_0, Q(T))$, where X_0 is the set of equivalence classes of R and $Q(T)$ is the decomposition topology on X_0, which is T_0 [11].

In [7] it was shown that every subspace of a R_1 space is R_1 and that (X, T) is R_1 iff $(X_0, Q(T))$ is T_2 and in [6] it was shown that the natural map P: $(X,T)\rightarrow(X_0, Q(T))$ is continuous, closed, open, and onto, and $P^{-1}(P(O))=O$ for all $O\in T$ and that if (X, T) is R_1, then $X_0=\{\overline{\{x\}}\mid x\in X\}$. These results can be used to obtain the following results.

THEOREM 2.2. *Every subspace of a strongly R_1 space is strongly R_1.*

DEFINITION 2.4. A subset A of (X, T) is *regular-open* iff $A=\text{Int }\overline{A}$. The set of all regular-open subsets of (X,T) forms a basis for a topology T_s on X, which is called the *semiregularization* of T [1].

THEOREM 2.3. *If (X, T) is a space, $\mathscr{B}=\{O\subset X\mid O$ is regular-open$\}$, and $\mathscr{B}_0=\{\mathscr{O}\subset X_0\mid \mathscr{O}$ is regular-open$\}$, then $\mathscr{B}_0=\{P(O)\mid O\in\mathscr{B}\}$.*

PROOF. Let $A\in\mathscr{B}$. Then $A=\text{Int }\overline{A}$ is open and $P^{-1}(P(A))=A$. Since P is continuous and open, then $P(A)=P(\text{Int }\overline{A})\subset\text{Int }P(\overline{A})\subset\text{Int }\overline{P(A)}$ and $P^{-1}(\text{Int }\overline{P(A)})\subset\text{Int }P^{-1}\overline{(P(A))}=\text{Int }\overline{P^{-1}(P(A))}=\text{Int }\overline{A}=A$, which implies $P(A)=\text{Int }\overline{P(A)}\in\mathscr{B}_0$. Let $\mathscr{O}\in\mathscr{B}_0$. Then $\mathscr{O}=\text{Int }\overline{\mathscr{O}}$. Since P is continuous and open, then $P^{-1}(\mathscr{O})=P^{-1}(\text{Int }\overline{\mathscr{O}})\subset\text{Int }P^{-1}(\overline{\mathscr{O}})=\text{Int }\overline{P^{-1}(\mathscr{O})}$ and $P(\text{Int }\overline{P^{-1}(\mathscr{O})})\subset\text{Int }P\overline{(P^{-1}(\mathscr{O}))}\subset\text{Int }\overline{P(P^{-1}(\mathscr{O}))}=\text{Int }\overline{\mathscr{O}}=\mathscr{O}$, which implies $P^{-1}(\mathscr{O})=\text{Int }\overline{P^{-1}(\mathscr{O})}\in\mathscr{B}$ and $\mathscr{O}=P(P^{-1}(\mathscr{O}))$.

In [10] it was shown that if (X, T) is strongly Hausdorff, then (X, T_s) is strongly Hausdorff. This result is combined with those above to obtain the following result.

THEOREM 2.4. *The following are equivalent*: (a) *(X, T) is strongly R_1,* (b) *$(X_0, Q(T))$ is strongly Hausdorff, and* (c) *(X, T_s) is strongly R_1 and $\overline{\{x\}}_{T_s}=\overline{\{x\}}_T$*

or all $x \in X$.

PROOF. (a) implies (b): Since (X, T) is strongly R_1, then (X, T) is R_1 and $(X_0, Q(T))$ is T_2, where $X_0=\{\overline{\{x\}} \mid x\in X\}$. Let $\{\overline{\{x_n\}}\}_{n\in N}$ be a sequence in X_0 such that $\overline{\{x_n\}}=\overline{\{x_m\}}$ iff $n=m$. Then there exists a sequence $\{U_n\}_{n\in N}$ of disjoint open sets in X such that $U_m \cap \{x_n \mid n\in N\}\neq\phi$ for all $m\in N$. Since P is open and $P^{-1}(P(O))=O$ for all $O\in T$, then $\{P(U_n)\}_{n\in N}$ is a sequence of disjoint open sets in X_0 and $P(U_m)\cap\{\overline{\{x_n\}} \mid n\in N\}\neq\phi$ for all $m\in N$.

(b) implies (c): Since $(X_0, Q(T))$ is strongly Hausdorff, then $(X_0, Q(T)_s)$ is strongly Hausdorff, (X, T) is R_1, and $X_0=\{\overline{\{x\}} \mid x\in X\}$. Then for each $\overline{\{x\}}\in X_0$, $\overline{\{\overline{\{x\}}\}}_{Q(T)_s}=\{\overline{\{x\}}\}$ and $X_0-\{\overline{\{x\}}\}=\bigcup_{\alpha\in A}\mathcal{O}_\alpha$, where $\mathcal{O}_\alpha$ is regular-open in X_0 for all $\alpha\in A$. Since $P^{-1}(P(O))=O$ for all $O\in T$, then by Theorem 2.3 $P^{-1}(\mathcal{O}_\alpha)$ is regular-open in X for all $\alpha\in A$ and $X-\overline{\{x\}}=P^{-1}(\bigcup_{\alpha\in A}\mathcal{O}_\alpha)=\bigcup_{\alpha\in A}P^{-1}(\mathcal{O}_\alpha)\in T_s$ and $\overline{\{x\}}_{T_s}\subset\overline{\{x\}}_T$. Since $T_s\subset T$, then $\overline{\{x\}}_T\subset\overline{\{x\}}_{T_s}$, which implies $\overline{\{x\}}_T=\overline{\{x\}}_{T_s}$. If $\{x_n\}_{n\in N}$ is a sequence in X such that $\overline{\{x_n\}}_{T_s}=\overline{\{x_m\}}_{T_s}$ iff $n=m$, then $\{\overline{\{x_n\}}\}_{n\in N}$ is a sequence in X_0 such that $\overline{\{x_n\}}=\overline{\{x_m\}}$ iff n=m and there exists a sequence $\{U_n\}_{n\in N}\subset Q(T)_s$ of disjoint sets such that $U_m\cap\{\overline{\{x_n\}} \mid n\in N\}\neq\phi$ for all $m\in N$ and $\{P^{-1}(U_n)\}_{n\in N}\subset T_s$ is a collection of disjoint sets such that $P^{-1}(U_m)\cap\{x_n \mid n\in N\}\neq\phi$ for all $m\in N$. By a similar argument, if $x, y\in X$ such that $\overline{\{x\}}_{T_s}\neq\overline{\{y\}}_{T_s}$, then there exist disjoint sets $U, V\in T_s$ such that $\overline{\{x\}}_{T_s}\subset U$ and $\overline{\{y\}}_{T_s}\subset V$.

Clearly (c) implies (a).

THEOREM 2.5. *For each $\alpha\in A$ let (X_α, T_α) be a topological space such that $X_\alpha\neq\phi$ and let S denote the product topology on $\prod_{\alpha\in A}X_\alpha$. Then $((\prod_{\alpha\in A}X_\alpha)_0, Q(S))$ is homeomorphic to $(\prod_{\alpha\in A}(X_\alpha)_0, W)$, where W is the product topology on $\prod_{\alpha\in A}(X_\alpha)_0$.*

PROOF. For each $C_{\prod_{\alpha\in A}\{y_\alpha\}}\in(\prod_{\alpha\in A}X_\alpha)_0$, $C_{\prod_{\alpha\in A}\{y_\alpha\}}=\prod_{\alpha\in A}C_{y_\alpha}$, where $C_{\prod_{\alpha\in A}\{y_\alpha\}}$ is the equivalence class in $\prod_{\alpha\in A}X_\alpha$ containing $\prod_{\alpha\in A}\{y_\alpha\}$ and C_{y_α} is the equivalence class in X_α containing y_α. Let $f=\left\{(C_{\prod_{\alpha\in A}\{y_\alpha\}}, \prod_{\alpha\in A}\{C_{y_\alpha}\}) \mid C_{\prod_{\alpha\in A}\{y_\alpha\}}\in(\prod_{\alpha\in A}X_\alpha)_0\right\}$. If $C_{\prod_{\alpha\in A}\{y_\alpha\}}=C_{\prod_{\alpha\in A}\{x_\alpha\}}$, then $\prod_{\alpha\in A}C_{y_\alpha}=\prod_{\alpha\in A}C_{x_\alpha}$, which implies $C_{y_\alpha}=C_{x_\alpha}$ for all $\alpha\in A$ and $\prod_{\alpha\in A}\{C_{y_\alpha}\}$

$=\prod_{\alpha\in A}\{C_{x_\alpha}\}$. Thus f is a function. Also, f is onto. If $C_y, C_x\in(\prod_{\alpha\in A}X_\alpha)_0$ such that $f(C_x)=f(C_y)$, then $\prod_{\alpha\in A}\{C_{x_\alpha}\}=\prod_{\alpha\in A}\{C_{y_\alpha}\}$, which implies $C_{x_\alpha}=C_{y_\alpha}$ for all $\alpha\in A$ and $C_x=C_y$. Thus f is $1-1$. For each $\alpha\in A$ let $f_\alpha:(X_\alpha, T_\alpha)\to((X_\alpha)_0, Q(T_\alpha))$ be the natural map and let $P:(\prod_{\alpha\in A}X_\alpha, S)\to((\prod_{\alpha\in A}X_\alpha)_0, Q(S))$ be the natural map. Let $\prod_{\alpha\in A}\mathscr{V}_\alpha\in W$ such that $\mathscr{V}_\alpha\in Q(T_\alpha)$ for all $\alpha\in A$ and $\mathscr{V}_\alpha=(X_\alpha)_0$ except for finitely many $\alpha\in A$. Then $f_\alpha^{-1}(\mathscr{V}_\alpha)=X_\alpha$ except for finitely many $\alpha\in A$ and since f_α is continuous, then $f_\alpha^{-1}(\mathscr{V}_\alpha)\in T_\alpha$, which implies $\prod_{\alpha\in A}f_\alpha^{-1}(\mathscr{V}_\alpha)\in S$. Since P is open, then $f^{-1}(\prod_{\alpha\in A}\mathscr{V}_\alpha)=P(\prod_{\alpha\in A}f_\alpha^{-1}(\mathscr{V}_\alpha))\in Q(S)$. Thus f is continuous. Let $\mathscr{O}\in Q(S)$. Let $\prod_{\alpha\in A}\{C_{y_\alpha}\}\in f(\mathscr{O})$ and let $y=\prod_{\alpha\in A}\{y_\alpha\}$. Then $y\in P^{-1}(\mathscr{O})\in S$ and there exists $\prod_{\alpha\in A}U_\alpha\in S$, where $U_\alpha\in T_\alpha$ for all $\alpha\in A$ and $U_\alpha=X_\alpha$ except for finitely many $\alpha\in A$, such that $y\in\prod_{\alpha\in A}U_\alpha\subset P^{-1}(\mathscr{O})$. Since $f_\alpha(U_\alpha)=(X_\alpha)_0$ except for finitely many $\alpha\in A$ and $C_{y_\alpha}\in f_\alpha(U_\alpha)\in Q(T_\alpha)$, then $f(C_y)=\prod_{\alpha\in A}\{C_{y_\alpha}\}\in\prod_{\alpha\in A}f_\alpha(U_\alpha)\in W$ and since $f^{-1}(\prod_{\alpha\in A}f_\alpha(U_\alpha))=P(\prod_{\alpha\in A}f_\alpha^{-1}(f_\alpha(U_\alpha)))=P(\prod_{\alpha\in A}U_\alpha)\subset\mathscr{O}$, then $\prod_{\alpha\in A}f_\alpha(U_\alpha)\subset f(\mathscr{O})$. Thus f is open.

In [10] it was shown that the product of nonempty topological spaces is strongly Hausdorff iff each coordinate space is strongly Hausdorff and in [4] it was shown that strongly Hausdorff is a topological property. These results can be combined with Theorem 2.4 and Theorem 2.5 to obtain the following result.

THEOREM 2.6. *The product of nonempty topological spaces is strongly R_1 iff each coordinate space is strongly R_1.*

In [7] it was shown that R_1 is a topological property. This result can be combined with a straightforward argument to obtain the following result.

THEOREM 2.7. *Strongly R_1 is a topological property.*

DEFINITION 2.5. A space is *rim-compact* iff each of its points has a base of neighborhoods with compact frontiers [11].

THEOREM 2.8. *Let (X, T) be rim-compact. Then the following are equivalent:* (a) *(X, T) is regular,* (b) *$(X_0, Q(T))$ is T_3,* (c) *$(X_0, Q(T))$ is Urysohn,* (d) *$(X_0, Q(T))$ is T_2,* (e) *(X,T) is R_1,* (f) *if $x, y\in X$ such that $\overline{\{x\}}\neq\overline{\{y\}}$, then there* *exist disjoint open sets U and V such that $x\in U$, $y\in V$, and $\overline{U}\cap\overline{V}=\phi$, and* (g) *$(X, T)$ is strongly R_1.*

PROOF. Clearly from the results above (a) implies (b) implies (c) implies (d) implies (e).

(e) implies (f): Let $x\in X$ and let $O\in T$ such that $x\in O$. Then there exists a neighborhood A of x such that $A\subset O$ and $\mathrm{Fr}(A)$ is compact. For each $y\in\mathrm{Fr}(A)$, $\overline{\{y\}}\neq\overline{\{x\}}$ and there exist disjoint open sets U_y and V_y containing x and y, respectively. For each $y\in\mathrm{Fr}(A)$, let U_y and V_y be disjoint open sets containing x and y, respectively. Then $\{V_y|y\in\mathrm{Fr}(A)\}$ is an open cover of $\mathrm{Fr}(A)$ and there exists a finite subcover $\{V_{y_i}|i=1, \cdots, n\}$. Then $x\in B=\left(\bigcap_{i=1}^{n} U_{y_i}\right)\cap(\mathrm{Int}(A))\in T$ and $\overline{B}\subset A\subset O$. Thus (X, T) is regular. If $a, b\in X$ such that $\overline{\{a\}}\neq\overline{\{b\}}$, then there exist disjoint open sets U and V such that $a\in U$ and $b\in V$ and since (X, T) is regular, there exist open sets W and Z such that $a\in W\subset\overline{W}\subset U$ and $b\in Z\subset\overline{Z}\subset V$.

(f) implies (g): Let $x\in X$ and let $O\in T$ such that $x\in O$. If $y\in X-O$, then $\overline{\{x\}}\neq\overline{\{y\}}$ and there exist disjoint open sets containing x and y, respectively, which implies $y\notin\overline{\{x\}}$ and $\overline{\{x\}}\subset O$. If $a, b\in X$ such that $\overline{\{a\}}\neq\overline{\{b\}}$, then there exist disjoint open sets U and V such that $a\in U$ and $b\in V$ and $\overline{\{a\}}\subset U$ and $\overline{\{b\}}\subset V$. Thus (X, T) is R_1 and $X_0=\{\overline{\{x\}}\,|\,x\in X\}$. Let $\overline{\{x\}}, \overline{\{y\}}\in X_0$ such that $\overline{\{x\}}\neq\overline{\{y\}}$. Then there exist disjoint open sets U and V in X such that $x\in U$, $y\in V$, and $\overline{U}\cap\overline{V}=\phi$. Then $P(U), P(V)\in Q(T)$ such that $\overline{\{x\}}\in P(U)$, $\overline{\{y\}}\in P(V)$ and $\overline{P(U)}\cap\overline{P(V)}=P(\overline{U})\cap P(\overline{V})=\phi$. Thus $(X_0, Q(T))$ is Urysohn, which implies $(X_0, Q(T))$ is strongly Hausdorff and (X, T) is strongly R_1.

(g) implies (a): Since (X, T) is R_1, then by the argument above (X, T) is regular.

3. Semi topological properties, minimal strongly R_1, and strongly R_1-closed

DEFINITION 3.1. Let (X, T) be a space and let $A\subset X$. Then A is *semi open* iff there exists $O\in T$ such that $O\subset A\subset\overline{O}$ [9].

DEFINITION 3.2. A 1—1 function from one space onto another space is a *semihomeomorphism* iff images of semi open sets are semi open and inverses of semi open sets are semi open. A property of topological spaces preserved by semihomeomorphisms is called a *semi topological property* [2].

In [3] it was shown that for a set X and a topology T on X, $[T]$, the equivalence class of topologies on X which yield the same semi open sets as T, has a finest element, denoted by $F(T)$, and $F(T)=\{O-N \mid O\in T$ and N is nowhere dense in $(X, T)\}$. In [2] and [4], respectively, it was shown that Hausdorff and strongly Hausdorff are semi topological properties. The following example shows that this result can not be extended to R_1 and strongly R_1.

EXAMPLE 3.1. Let T be the usual topology on N. Then $(\beta N, W)$, the Stone-Čech compactification of (N, T), is extremely disconnected and has non isolated points [11]. Let x be a non isolated point of βN and let $y \notin \beta N$. Then $S=\{O\in W \mid x\notin O\} \cup \{O\cup\{y\} \mid x\in O\in W\}$ is a topology on $Y=\beta N\cup\{y\}$ and (Y, S) is regular, which implies (Y, S) is strongly R_1. The identity function from (Y, S) onto $(Y, F(S))$ is a semihomeomorphism. Since $\overline{\{x\}}_{F(S)}=\{x\}\neq\{y\}=\overline{\{y\}}_{F(S)}$ and there do not exist disjoint elements of $F(S)$ containing x and y, respectively, then $(Y, F(S))$ is not R_1.

DEFINITION 3.3. A space (X, T) with property P is called *minimal* P iff X has no strictly courser P-topologies [10].

In [10] minimal strongly Hausdorff was investigated and characterized. Since each set X with the indiscrete topology is R_1 and strongly R_1, then (X, T) is minimal R_1 or minimal strongly R_1 iff T is the indiscrete topology on X.

DEFINITION 3.4. A space (X, T) with property P is called *P-closed* iff X is a closed subspace in every P-space in which it is embedded [10].

In [10] strongly Hausdorff-closed was investigated and characterized. The last result investigates R_1-closed and strongly R_1-closed.

THEOREM 3.1. $\{(X, T) \mid (X, T)$ *is* R_1*-closed or strongly* R_1*-closed*$\}=\phi$.

The proof follows by using a construction similar to that in Example 3.1 and is omitted.

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