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ALMOST POINTWISE PERIODIC SEMIGROUPS II*

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The author investigated a structure theorem of an almost pointwise periodic semigroup on an arc [1]. The purpose of this paper is to give new proofs of the following theorems:

(1) If H is a subset of a continuum semigroup S with non-empty boundary $F(H)$ and if the closure H^* of H contains a point x in S such that $Sx \subset H^*$, then $Sp \subset H^*$ for some $p \in F(H)$ [3].

(2) Every almost pointwise periodic standard thread is a semilattice [1].

A *topological semigroup* is a Hausdorff space S together with a continuous function $S \times S \to S$ (whose value at (x,y) will be denoted by xy) satisfying

$$(xy)z = x(yz)$$

for all x, y, z in S [2] [6].

Throughout, a semigroup will mean a topological semigroup. Let S be a semigroup and let $A \subset S$. Then $L_0(A)$ denotes the union of all left ideals of S contained in A. If A contains no left ideal of S, then $L_0(A)=\phi$. If $L_0(A) \neq \phi$, then it is clearly the unique largest left ideal of S contained in A.

LEMMA 1. *Let S be a semigroup and let $A \subset S$.*

(1) *If A is closed, then $L_0(A)$ is closed.*

(2) *If A is open and S is compact, then $L_0(A)$ is open* [6].

PROOF. (1) Since $L_0(A)$ is a left ideal of S, $L_0(A)^*$ is also a left ideal of S. By definition, $L_0(A) \subset A$ and hence $L_0(A)^* \subset A^* \subset A$. Therefore $L_0(A)^* = L_0(A)$.

(2) Let $x \in L_0(A)$. Then $Sx \subset SL_0(A) \subset L_0(A)$. Since S is compact and since A is open, there is an open set U about x such that $SU \subset A$. Let $B = U \cup SU$. Then B is the ideal of S generated by U and $B \subset A$. Therefore

$$x \in U = U^\circ \subset B \subset J_0(A)$$

and $J_0(A)$ is open.

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THEOREM 2. *Let S be a continuum semigroup. If H is a subset of S with non-empty boundary $F(H)$ and if the closure H^* of H contains a point x in S such that $Sx \subset H^*$, then $Sp \subset H^*$ for some $p \in F(H)$* [3].

PROOF. Let $x \in H^*$ such that $Sx \subset H^*$. Since $H^* = H^\circ \cup F(H)$, $x \in H^\circ$ or $x \in F(H)$. If $x \in F(H)$, then we are done. Suppose $x \in H^\circ$. If $Sx \cap F(H) \neq \phi$, then there exists an element $t \in S$ such that $tx \in F(H)$. Let $p = tx$. Then $Sp = Stx \subset Sx \subset H^*$.

Now suppose $Sx \cap F(H) = \phi$. Then $Sx \subset H^\circ$. Since Sx is a left ideal of S, $L_0(H^\circ) \neq \phi$ and $L_0(H^\circ)$ is open by Lemma 1. If $L_0(H^\circ)^* \subset H^\circ$, by the definition of $L_0(H^\circ)$, $L_0(H^\circ)^* = L_0(H^\circ)$. Then $L_0(H^\circ)$ is a proper clopen subset of S which contradicts the fact that S is connected. Hence we have $L_0(H^\circ)^* \cap F(H) \neq \phi$. Let $p \in L_0(H^\circ)^* \cap F(H)$. Then $p \in F(H)$ and

$$Sp \subset SL_0(H^\circ)^* \subset L_0(H^\circ)^* \subset H^*.$$

An *arc* is a continuum with exactly two non-cutpoints. It is well known that any arc admits a total order and has one non-cutpoint as a least element and the other non-cutpoint as a greatest element [7]. It is supposed that an arc to have such a total order on it. We will denote an arc with endpoints a and b, $a < b$, by $[a, b]$ and if $x, y \in [a, b]$, $x < y$, then $[x, y] = \{t \mid x \leqq t \leqq y\}$.

A *standard thread* is a semigroup on an arc in which the greatest element is an identity and the least element is a zero.

LEMMA 3. *Suppose $S = [z, u]$ is a standard shread. Then*
(1) *$xS = Sx = [z, x]$ for all x in S.*
(2) *$x \leq y$ and $v \leq w$ imply $xv \leq yw$* [6].

PROOF. (1) Let $H = [z, x]$, $x < u$. Then $F(H) = \{x\}$. Since $Sz = \{z\} \subset H = H^*$ and $z \in H^*$, by Theorem 2, $Sx \subset H^* = H$. Since z, $x \in Sx$ and Sx connected, $H = [z, x] \subset Sx$. Hence we have

$$Sx = xS = [z, x], \ \forall x \in S.$$

(2) Since $x \leq y$, $x \in [z, y] = Sy$ and $xv \in Syv = [z, yv]$. Hence $xv \leq yv$. Again, since $v \leq w$, $v \in [z, w] = wS$ and $yv \in ywS = [z, yw]$. Hence $yv \leq yw$. Therefore $xv \leq yw$.

A semigroup S is termed *almost pointwise periodic at* $x \in S$ iff for each open set U about x, there is an integer $n > 1$ such that $x^n \in U$. S is said to be *almost pointwise periodic* iff S is almost pointwise periodic at every $x \in S$ [4] [5].

LEMMA 4. *Let K be a compact subsemigroup of a semigroup S. Then S is not almost pointwise periodic at every point of $K-K^2$* [1] [4].

THEOREM 5. *Every almost pointwise periodic standard thread is a semilattice* [1].

PROOF. Let $S=[z,u]$ be an almost pointwise periodic standard thread and let $p\in S$ with $p\neq z$, u. Suppose $p^2\neq p$. Then $p^2<p$ and $p\in[z,p]-[z,p^2]=[z,p]-[z,p]^2$ by Lemma 3. Then by Lemma 4, S is not almost pointwise periodic at p which is a contradiction. Hence we have $p^2=p$. Since every standard thread is commutative, S is semilattice.

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