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BOUNDARY VALUE PROBLEMS FOR LINEAR OPERATORS

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1. Introduction

Let H and K be Hilbert spaces, and let $H_0' \subset H'$ be closed linear submanifolds of H. Let P be the orthogonal projection from H onto H_0', and let C be a bounded linear operator from H into K such that $CC^*=I$, the identity, on K. Let $g \in K$ and $z \in H$ be given. Consider the problem of finding $u \in H'$ such that

$$\text{(BVP)}\ Cu=g,\ (P-I)u=z.$$

According to J. W. Neuberger [5], this problem has a direct application to a wide class of ordinary or partial functional differential equations. He gave in [5] several sufficient conditions for (BVP) to have a solution or a unique solution in the special case when

$$(1.1) \qquad (\text{Range}C^*) \cap (H \ominus H_0') = \{0\}.$$

In the case when a certain iteration converges and (1.1) holds, he also showed how to find explicitly a solution of the problem.

While his method is constructive, the assumption (1.1) is to too restrictive. Therefore it is the purpose of this note to consider a more wide (and natural) class of (BVP) which contains (1.1) as a special case. While our method is nonconstructive, our method is very elementary and makes use of adjoints only. The present note grew out with a conversation with J. W. Neuberger who suggested that [5] and [3] might have a connection. But it turned out that there is no direct one because the condition $(P-I)u=z$ is not a boundary condition in the sense of [3].

2. Results

Define two operators T_1 and C_0 by

$$T_1u=\{Cu,\ (P-I)u\},\ u \in \text{Domain } T_1 \equiv H',$$

$$C_0u=Cu,\ u \in \text{Domain } C_0 \equiv H_0'.$$

Thus $G(T_1) \subset H \oplus (K \oplus H)$, $G(C_0) \subset H \oplus K$, where $G(T_1)$ denotes the graph of T_1.

LEMMA 1. (I) *The following* (I-1)—(I-3) *are equivalent*:

(I-1) *T_1 has a closed range in $K\oplus H$.*

(I-2) (Range C^*)+$(H\ominus H_0')$ *is closed in H, where + denotes an algebraic sum.*

(I-3) *C_0 has a closed range in K.*

(II) $(\mathrm{Null}(G(T_1))^*)^{\perp}=\{\{u, v\}\in K\oplus H \mid v\in H'\ominus H_0',\ C^*u+v\in((\mathrm{Range}C^*)\cap (H\ominus H_0'))^{\perp}\}$, *where the adjoint $(G(T_1))^*$ (see* [1] *or* [2]*) of $G(T_1)$ is taken in $(K\oplus H)\oplus H$.*

(III) $(\mathrm{Range}C^*)\cap(H\ominus H_0')=\{0\}$ *if, and only if* $\mathrm{Null}(PC^*)=\{0\}$.

(IV) *The following* (IV-1)—(IV-3) *are equivalent*:

(IV-1) *The solution of* (BVP), *if exists, is unique.*

(IV-2) (Range C^*)+$(H\ominus H_0')$ *is dense in H.*

(IV-3) *C_0 is one-to-one.*

PROOF. We can compute easily that

(2.1) $$(G(T_1))^*=\{\{\{x, y\},\ v\}\in(K\oplus H)\oplus H \mid C^*x+(P-I)y-v\in H\ominus H'\},$$

(2.2) $$(G(C_0))^*=\{\{x, y\}\in K\oplus H \mid C^*x-y\in H\ominus H_0'\},$$

where the second adjoint is taken in $K\oplus H$.

It follows that

(2.3) $$\mathrm{Null}(G(T_1))^*=\{\{x, y_1+y_2+y_3\}\in K\oplus H \mid\ y_1\in H_0',\ y_2\in H'\ominus H_0',\ y_3\in H\ominus H' \text{ such that } C^*x-y_2\in H\ominus H'\},$$

(2.4) $$\mathrm{Range}(G(T_1))^*=\mathrm{Range}(G(C_0))^* =(\mathrm{Range}\ C^*)+(H\ominus H_0'),$$

(2.5) $$\mathrm{Null}(G(C_0))^*=\{x\in K \mid C^*x\in H\ominus H_0'\}.$$

Now, T_1 has a closed range if, and only if $(G(T_1))^*$ has a closed range (Theorem 2.3, [1]).

Thus (I) is immediate by (2.4).

(II) $\{u, v\}\in(\mathrm{Null}(G(T_1))^*)^{\perp}$ if, and only if $0=(u, x)+(v, y_1+y_2+y_3)$ for all $\{x, y_1+y_2+y_3\}\in K\oplus H$ satisfying the conditions in the right of (2.3). Here (,) denotes the inner product in H or K. Since C^* is an isometry into H, it follows that

$$0=(C^*u, y)+(v, y)$$

for all $y\in(\mathrm{Range}\ C^*)\cap(H\ominus H_0')$.

Thus $\{u, v\}$ belongs to the set in the right of (II).

(III) This is clear as C^* is one-to-one.

(IV) Clearly (IV-1) holds if, and only if

$$\{0\}=\mathrm{Null}T_1=(\mathrm{Range}(G(T_1))^*)^{\perp}.$$

Using (2.4), this is equivalent to (IV-2) and (IV-3). This completes the proof.

THEOREM 2. (I) *If* (BVP) *has a solution, then*

$$z\in H'\ominus H_0',$$

$$C^*g+z\in((\mathrm{Range}\ C^*)\cap(H\ominus H_0'))^{\perp}.$$

(II) *If $C(H_0')$ is closed in K, then the converse of* (I) *holds.*

(III) *Assume that $C(H_0')$ is closed. Then* (BVP) *has a unique solution if, and only if*

$$z\in H'\ominus H_0',$$

$$C^*g+z\in((\mathrm{Range}\ C^*)\cap(H\ominus H_0'))^{\perp},$$

and C_0 is one-to-one.

PROOF. (I) If (BVP) has a solution in H', then

$$\{g,\ z\}\in(\mathrm{Range}\ T_1)^c=(\mathrm{Null}(G(T_1))^*)^{\perp}.$$

Thus the result follows from (II) Lemma 1.

(II) By (I) Lemma 1, Range T_1 is closed. Thus by (II) Lemma 1,

$$\{g,\ z\}\in(\mathrm{Range}\ T_1)^c=\mathrm{Range}\ T_1.$$

(III) This is clear by the above two parts and (IV) Lemma 1. This completes the proof.

REMARK. By Theorem 2.1, [1],

$$((\mathrm{Range}\ C^*)\cap(H\ominus H_0'))^{\perp}=(\mathrm{Null}\ C)+H_0'.$$

Provided that Null $C+H_0'$ is closed.

COROLLARY 3 (Theorem 2, [5]). *Let $g\in K$, $w\in H'$. If C_0 is one-to-one and onto K, then there exists a unique $u\in H'$ such that*

$$Cu=g,\ Pu-w=u-w.$$

PROOF. Since $C_0(H_0')=K$,

$$\{0\}=\mathrm{Null}(G(C_0))^*=\mathrm{Null}(PC^*).$$

Thus by (III) of Lemma 1,

$$(\text{Range } C^*) \cap (H \ominus H_0') = \{0\}.$$

Now apply (Ⅲ) of Theorem 2.

REMARK. The closedness condition in (Ⅲ) Theorem 2 can be replaced by a different condition. For example in (Theorem 3, [5]), it is proved that if (Ⅰ-1) is satisfied and the limit of a certain iteration exists, then (BVP) with $z=Pw-w$ has a solution in H'.

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