Kyungpook Math. J.
Volume 21, Number 2
December, 1981

ON REGULAR AND CONTINUOUS RINGS II

By Roger Yue Chi Ming

1. Introduction

This is a natural sequel to [8] and [14]. GLD (generalized left duo) and ALD (almost left duo) rings are introduced in [8] and [9] respectively. In [14], the following generalization is considered: A is called a (left) CM-*ring* if, for any maximal essential left ideal M of A, every complement left subideal is an ideal of M. Working along the lines suggested by the referee of [14], we now look more closely at the connections between GLD, ALD and CM-rings in the first section of this note. In so far as Von Neumann regular, left and right V-rings are concerned, our attempt will be most satisfactory (several of our previous results are here extended). Next, we introduce weak Up-injective rings to continue the study of continuous regular rings in [8].

Throughout, A represents an associative ring with identity and A-modules are unitary. Z will denote the left singular ideal of A. For completeness, recall that (1) A left A-module M is *p-injective* if, for any principal left ideal P of A and any left A-homomorphism $g: P \to M$, there exists $y \in M$ such that $g(b)=by$ for all $b \in P$; (2) A is a *(left)* WP-*ring* (*weak p-injective*) if every left ideal not isomorphic to ${}_AA$ is p-injective. Von Neumann regular rings may be characterized by any one of the following conditions: (a) Every left A-module is flat; (b) Every left A-module is p-injective. As pointed out in [13], if I is a p-injective left ideal of A, then A/I is a flat left A-module. It is now well-known that there is no inclusion relation between the classes of regular rings and V-rings (this has motivated the study of various connections between regular rings, V-rings and related rings (cf. for example, [1] and [3])).

2. CM and regular rings

Apart from generalizing GLD and ALD rings, CM-rings (introduced in [14]) include left Ore domains and left PCI rings studied by A. K. Boyle, C. Faith and R. F. Damiano (cf. [2]). Our first result shows that if A is von Neumann regular, then A is CM iff A is ALD iff A is GLD (which answers a query due to the referee of [14] and also improves [14, Remark 7]). However, this is not true for V-rings. Indeed, CM left (or right) V-rings need not be regular (the domains constructed by J. H. Cozzens are relevant examples). We first continue the study of CM-rings. As usual, (a) an ideal of A means a two-sided ideal and (b) a left (right) ideal is called reduced if it contains no non-zero nilpotent element. The next lemma improves [14, Lemma 1.6, Theorem 1.9, Lemma 2.1, Theorem 2.2(2) and Proposition 2.4].

LEMMA 1.1. *If A is a left non-singular CM-ring, then A is either semi-simple Artinian or reduced.*

PROOF. Suppose A is not semi-simple Artinian. Then there exists a maximal left ideal M of A which is essential. By [14, Lemma 1.6(1)], M is reduced. If $0\neq b\in A$ such that $b^2=0$, since $0\neq ab\in M$ for some $a\in A$, then $(bab)^2=0$ and $bab \in M$ together imply $bab=0$. It follows that $(ab)^2=0$ which implies $ab=0$, contradicting $ab\neq 0$. This proves that A is reduced.

The next result then follows immediately.

THEOREM 1.2. *The following conditions are equivalent:*

(1) *A is either semi-simple Artinian or strongly regular.*

(2) *A is a left non-singular, left p-injective CM-ring.*

(3) *A is a left non-singular, right p-injective CM-ring.*

(4) *A is a left non-singular CM-ring whose simple left modules are flat.*

(5) *A is a left non-singular CM-ring whose simple right modules are flat.*

Applying [8, Proposition 2.1] to Lemma 1.1, we have

PROPOSITION 1.3. *If A is a left non-singular CM-ring, then the maximal left quotient ring of A coincides with the right one.*

Now write "A is ECM" if, for any maximal essential left ideal M of A, every complement or essential left subideal is an ideal of M. The next lemma improves [13, Lemma 1.1].

LEMMA 1.4. *If A is semi-prime ECM-ring, then A is either semi-simple Artinian or reduced.*

PROOF. We see from Lemma 1.1 that it is sufficient to prove that $Z=0$. Suppose the contrary: let $0\neq z\in Z$ such that $z^2=0$. If M is a maximal left ideal containing $l(z)$, then $l(z)M\subseteq l(z)$ implies $(Mz)^2\subseteq AzMz\subseteq l(z)Mz=0$ whence

$M=l(z)$ (since A is semi-prime). Therefore $Az(\approx A/M)$ is a minimal left ideal and hence a direct summand of ${}_AA$ which implies $z=0$, a contradiction.

Following [6], a left A-module M is called *semi-simple* if the intersection of the maximal left submodules of M is zero. In [6, Theorem 2.1], right V-rings are characterized in terms of semi-simple right modules and intersections of maximal right ideals. The next theorem partially extends [13, Theorem 1.3].

THEOREM 1.5. *The following conditions are equivalent for an ECM ring A:*

(1) *A is regular.*

(2) *A is a left or right V-ring.*

(3) *A is fully left or right idempotent.*

(4) *Every cyclic semi-simple left A-module is flat.*

(5) *A is a semi-prime ring whose principal left ideals are complement left ideals.*

(6) *Any proper left ideal of A which contains all the minimal projective left ideals is an intersection of maximal left ideals.*

(Use [10, Theorem 1], Lemma 1.4 and the proof of [13, Theorem 1.3].)

Since ECM rings still generalize ALD and GLD rings, the next corollary then follows.

COROLLARY 1.6. *If A is either regular or a left or right V-ring, then A is GLD iff A is ALD iff A is ECM.*

(cf. [8], [9] and [13]).

Rings whose left ideals not isomorphic to ${}_AA$ are quasi-injective, called *left wq-rings*, are studied in [7]. Principal ideal domains are noted PID.

Applying [7, Lemma 1.5] and [12, Corollary 1.6] to Theorem 1.2, we get

PROPOSITION 1.7. (1) *A CM, WP-ring is either semi-simple Artinian or strongly regular or a left PID.*

(2) *A semi-prime CM, left wq-ring is either semi-simple Artinian or left and right self-injective strongly regular or a left PID.*

We add a remark on wq-rings.

REMARK 1. (a) A is simple Artinian iff A is a prime unit-regular left wq-ring; (b) A prime left wq, left and right V-ring is either Artinian or a simple left PID.

Since left Ore domains are CM-rings, it is natural to ask: when is a prime CM-ring a left Ore domain?

PROPOSITION 1.8. *The following conditions are equivalent*:

(1) *A is a left Ore domain.*

(2) *A is a prime CM-ring containing a non-zero reduced left ideal.*

PROOF. Obviously, (1) implies (2). Assume (2). Let I be a non-zero reduced left ideal. If ${}_AI$ is essential in ${}_AA$, then A is an integral domain [11, Proposition 6]. If not, then $I\oplus K$ is an essential left ideal for some non-zero complement left ideal K of A. Suppose $I\oplus K\neq A$. If M is a maximal left ideal containing $I\oplus K$, then $KI\subseteq KM\subseteq K$ implies $KI\subseteq K\cap I=0$, contradicting the primeness of A. Thus $I\oplus K=A$ which implies A an integral domain again [11, Proposition 6]. The preceding argument also shows that any non-zero left ideal of A must be essential which proves that (2) implies (1).

REMARK 2. (a) If A is a CM-ring, then a minimal left ideal is injective iff it is p-injective. (b) A CM-ring is a left V-ring iff it is a left V-ring [11]. (c) If A is an ECM-ring whose singular left modules are injective, then A is either semi-simple Artinian or strongly regular left hereditary (this generalizes the corresponding commutative case studied by V.C. Cateforis and F.L. Sandomierski).

3. WUP and continuous regular rings

Recall that A is *left continuous* (in the sense of Y. Utumi) if (a) every left ideal isomorphic to a direct summand of ${}_AA$ is itself a direct summand of ${}_AA$ and (b) every complement left ideal of A is a direct summand of ${}_AA$. As from now on, we shall call a left A-module M *Up-injective* (*Utumi p-injective*) if, for any complement left ideal C of A, $a\in A$, any left A-homomorphism $g: Ca\to M$, there exists $y\in M$ such that $g(ca)=cay$ for all $c\in C$. It then follows that a complement left ideal which is Up-injective is generated by an idempotent. Rings whose proper complement left ideals are Up-injective are therefore the left CS-rings studied by Chatters-Hajarnavis [5].

We now introduce left weak Up-injective rings.

DEFINITION. A is called a *(left) WUP ring* if every left ideal not isomorphic to ${}_AA$ is Up-injective.

Obviously, a WUP ring is WP. On the other hand, WUP rings generalize left PID and left continuous regular rings.

LEMMA 2.1. *Let A be a WUP ring. Then*

(1) *A is a semi-prime ring whose finitely generated and complement left ideals are principal projective.*

(2) *For any idempotent e, either Ae or A(1-e) is Up-injective. Consequently, for any $a \in A$, either Aa or $l(a)$ is Up-injective.*

PROOF. (1) If C is a complement left ideal not isomorphic to ${}_AA$, then ${}_AC$ is Up-injective which implies ${}_AC$ a direct summand of ${}_AA$. Then every complement left ideal is principal and (1) follows from [12, Lemma 1.1].

(2) Since a left A-module isomorphic to a Up-injective left A-module is Up-injective while a direct summand of a Up-injective left A-module is Up-injective, then (2) is proved as in [12, Lemma 1.8].

We are now in a position to give some characteristic properties of left continuous regular rings.

THEOREM 2.2. *The following conditions are equivalent:*

(1) *A is left continuous regular.*

(2) *Every left A-module is Up-injective.*

(3) *A is a WUP ring such that the square of every principal left ideal is a left annihilator.*

(4) *A is a WUP ring such that the square of every principal right ideal is a right annihilator.*

(5) *A is a WUP ring which is either left or right p-injective.*

(6) *A is a WUP ring whose principal left ideals are complement left ideals.*

(7) *Every complement left ideal of A is principal and every cyclic singular left A-module is flat.*

(8) *A is a left non-singular ring whose principal and complement left ideals coincide such that any cyclic non-singular left A-module is flat.*

(9) *A is left non-singular such that for any non-singular left A-module M, $l(y)$ is Up-injective for every $y \in M$.*

PROOF. Since Up-injectivity coincides with p-injectivity over a left continuous regular ring, then (1) implies (2).

Since Up-injectivity implies p-injectivity in general, then (2) implies (3), (4), (6) and (9).

Assume (3). For any $b\in A$, $r((Ab)^2)=r(Ab)$ since A is semi-prime and $Ab\subseteq l(r(Ab))=l(r((Ab)^2))=(Ab)^2$ implies $Ab=(Ab)^2$ is a left annihilator. A theorem of M. Ikeda-T. Nakayama then asserts that A is right p-injective which shows that (3) implies (5).

Similarly, (4) implies (5).

Since a WP-ring which is either left or right p-injective is von Neumann regular, then (5) implies (1) by Lemma 2.1.

(6) implies (7): For any complement left ideal C, if K is a relative complement such that $C\oplus K$ is essential, then $C\oplus K$ is finitely generated and hence principal (Lemma 2.1) which implies that $C\oplus K$ is a complement left ideal. Therefore $C\oplus K=A$ which proves A regular.

Since A is regular iff every cyclic singular left A-module is flat, then (7) implies (8).

(8) implies (1): Since $Z=0$, for any complement left ideal C of A, A/C is a non-singular left A-module which is therefore flat. Since C is principal, then ${}_AA/C$ is finitely related which implies ${}_AA/C$ projective whence ${}_AC$ is a direct summand of ${}_AA$.

Finally assume (9). Then A is left p-injective left non-singular. Let $M=Au$ be a cyclic non-singular left A-module. If ${}_AI$ is an essential extension of $l(u)$ in ${}_AA$, for any $a\in I$, there exists an essential left ideal L such that $La\subseteq l(u)$ which implies $au\in Z(M)$, the singular submodule of M. Since $Z(M)=0$, then $a\in l(u)$ which proves that $l(u)=I$ is a complement left ideal of A. Then l(u) Up-injective implies $l(u)$ a direct summand of ${}_AA$ whence A^M is projective. Therefore A is a left p-injective ring whose principal left ideals are projective which implies A regular. If C is a complement left ideal of A, since $Z=0$, then ${}_AA/C$ is non-singular which implies ${}_AA/C$ projective. Thus ${}_AC$ is a direct summand of ${}_AA$ which proves that A is left continuous and therefore (9) implies (1).

The next corollary then follows from [12, Lemma 1.3, Theorem 1.5 and Corollary 1.6], Theorem 1.2, Lemma 1.4 and Theorem 2.2.

COROLLARY 2.3. (1) *A WUP ring is either indecomposable or left continuous regular*;

(2) *A directly finite WUP ring is either a left PID or left continuous regular*;

(3) *A reduced WUP ring is either a left PID or a continuous strongly regular*

ring;

(4) *A semi-prime ECM ring whose left annihilator ideals are Up-injective is either semi-simple Artinian or continuous strongly regular*;

(5) *A CM-ring whose essential left ideals are Up-injective is either semi-simple Artinian or continuous strongly regular.*

Applying [12, Proposition 1.9] to Theorem 2.2, we get

COROLLARY 2.4. *Let A be a WUP ring. Then A is left continuous regular if any one of the following conditions is satisfied*:

(1) *A contains a central zero-divisor.* (2) *The left socle of A is non-zero finitely generated.* (3) *A is a direct sum of two left ideals which are of infinitele ft Goldie dimension.*

It is known that a right PCI ring is either semi-simple Artinian or a simple right Noetherian right hereditary V-domain [2, Theorem 1].

REMARK 3. A WUP, right PCI ring is left PCI.

Call A an ELT (resp. MELT) *ring* if every essential (resp. maximal essential) left ideal is an ideal of A(cf. [11]).

REMARK 4. A is ELT left continuous regular iff A is a MELT, WUP ring whose simple right modules are flat.

REMARK 5. If A is MELT, then A is von Neumann regular iff for any maximal left ideal M and any $a \in A$, ${}_A A/Ma$ is p-injective and flat.

REMARK 6. If A is an ELT ring whose factor rings are semi-prime and whose primitive factor rings are regular, then A is regular. (This is related to [3, Problems 1 and 4] and improves [10, Proposition 6].)

REMARK 7. If A is commutative and P a non-singular ideal generated by an element, then P is generated by an idempotent iff p^2 is an annihilator.

Université Paris VII
U.E.R. de Mathématiques
2, Place Jussieu
75251 Paris cedex 05
France

REFERENCES

[1] E.P. Armendariz and J.W. Fisher, *Regular P.I.-rings*, Proc. Amer. Math. Soc. 39(1973), 247—251.

[2] R.F. Damiano, *A right PCI ring is right Noetherian*, Proc. Amer. Math. Soc. 77(1979), 11—14.

[3] J.W. Fisher, *Von Neumann regular rings versus V-rings, Ring Theory*, Proc. Oklahoma Conference, Lecture notes n° 7, Dekker (New York) (1974), 101—119.

[4] K.R. Goodearl, *Von Neumann regular rings*, Monographs and studies in Maths., 4, Pitman (London) (1979).

[5] A.W. Chatters and C.R. Hajarnavis, *Rings in which every complement right ideal is a direct summand*, Quart. J. Math. Oxford (2) 28 (1977), 61—80.

[6] G.O. Michler and O.E. Villamayor, *On rings whose simple modules are injective*, J. Algebra 25 (1973), 185—201.

[7] S. Mohamed and S. Singh, *Weak q-rings*, Canad. J. Math. 29 (1977), 687—695.

[8] R. Yue Chi Ming, *On regular rings and continuous rings*, Math. Japonica 24 (1979), 563—571.

[9] R. Yue Chi Ming, *On regular rings and V-rings*, Monatshefte für Math. 88 (1979), 335—344.

[10] R. Yue Chi Ming, *On von Neumann regular rings*, IV, Riv. Mat. Univ. Parma (4) 6 (1980), 181—188.

[11] R. Yue Chi Ming, *On V-rings and prime rings*, J. Algebra 62 (1980), 13—20.

[12] R. Yue Chi Ming, *Von Neumann regularity and weak p-injectivity*, Yokohama Math. J. 28(1980), 61—70.

[13] R. Yue Chi Ming, *On von Neumann regular rings*, V, Math. J. Okayama Univ. 22 (1980), 151—160.

[14] R. Yue Chi Ming, *On regular rings and self-injective rings*, Monatshefte für Math. 91(1981), 201—215.