Kyungpook Math. J.
Volume 21, Number 2
December. 1981

CHARACTERIZATIONS OF IRREDUCIBLE SPACES

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In this paper, we briefly investigate a certain class of non-Hausdorff spaces called irreducible spaces. A topological space (X, T) is irreducible if and only if every non-empty $V\in T$, $W\in T$, $V\cap W\neq\phi$ [7]. The following is an example of irreducibility as it arises in the study of algebraic geometry.

EXAMPLE 1. Let A be a commutative ring with identity 1, and let Spec(A) denote the set of all prime ideals of A. For $E\subset A$, define $V(E)=\{x\in\text{Spec}(A)\mid E$ is contained in the ideal $x\}$. Then $\{V(E)\mid E\subset A\}$ satisfies the axioms for closed sets in a topology on Spec(A). Call this topology T. Let N be the ideal of nilpotent elements in A. Then N is prime if and only if A/N is an integral domain if and only if (Spec(A), T) is irreducible [7, pp. 17—21].

DEFINITION 2. A set A in a topological space (X, T) is *semiopen* if and only if there exists a $V\in T$ such that $V\subset A\subset\overline{V}$, where $\overline{V}$ is the closure of V.

DEFINITION 3. A function $f: X\rightarrow Y$ is said to be *irresolute* (*semi-continuous*) if and only if the inverse image of every semi-open (open) set is semi-open.

DEFINITION 4. A topological space (X,T) is *S-closed* if and only if for every semi-open cover $\{U_a\mid a\in A\}$ of X there exists a finite subfamily such that the union of their closures cover X [8].

DEFINITION 5. A topological space (X, T) is *S-connected* if and only if X cannot be written as the disjoint union of two non-empty semi-open sets.

DEFINITION 6. A topological space (X, T) is an *S-continuum* if and only if X is S-closed and S-connected.

It is immediate from the definition that an irreducible space is connected(in the usual sense) and non-Hausdorff. As shown by Example 7, an irreducible space need not be compact.

EXAMPLE 7. The real number line R with the co-finite topology is irreducible.

EXAMPLE 8. The real number line with the co-countable topology is irreducible.

EXAMPLE 9. The real number line R with the right ray topology is irreducible.

EXAMPLE 10. For a polynomial P in n real variables, let $Z(P)=\{\bar{x}\in R^n \mid P(\bar{x})=0\}$. Let $\mathscr{P}$ be the collection of all such polynomials. Then $\{Z(P) \mid p\in\mathscr{P}\}$ is a base for the closed sets of a topology on R^n that is irreducible. This topology is commonly called the *Zariski topology*.

THEOREM 11. *$\prod_{a\in A} X_a$ is irreducible if and only if X_a is irreducible for every $a\in A$.*

PROOF. Let $V=(V_{a_1}\times \prod_{a\neq a_1} X_a)$ and $W=(W_{a_2}\times \prod_{a\neq a_2} X_a)$ be two arbitrary subbasic open sets. If $a_1\neq a_2$, then $V\cap W=(V_{a_1}\times W_{a_2})\times \prod_{a\neq a_1, a_2} X_a\neq\phi$. If $a_1=a_2$, then $V\cap W=(V_{a_1}\cap W_{a_2})\times \prod_{a\neq a_1} X_a$. But since $X_{a_1}=X_{a_2}$ is an irreducible space, we have $(V_{a_1}\cap W_{a_2})\neq\phi$, which implies that $V\cap W\neq\phi$. Therefore, $\prod_{a\in A} X_a$ is an irreducible space.

Conversely, assume $\prod_{a\in A} X_a$ is an irreducible space. The projection map $\Pi_b\colon \prod_{a\in A} X_a\to X_b$ is a continuous surjection. Therefore, by Proposition 2.1(v) [7], X_b is an irreducible space for every $b\in A$.

THEOREM 12. *Let X be irreducible. If $f\colon X\to Y$ is a continuous function with closed graph, then f is constant.*

PROOF. Let $x\in X$ be chosen arbitrarily. Define the spiral of x, denoted $\mathrm{Sp}(x)$, by $\mathrm{Sp}(x)=\bigcap\{\bar{U} \mid U\in N(x)\}$ [10]. By Theorem 7 of [10], f is constant on $\mathrm{Sp}(x)$. Therefore, since every open set in an irreducible space is dense [7, p. 13], we have $\mathrm{Sp}(x)=X$ and f is constant.

THEOREM 13. *If X is irreducible and $f\colon X\to Y$ is a semi-continuous surjection, then Y is irreducible.*

PROOF. Suppose that Y is not irreducible. Then there exists open sets V, W in Y such that $V\cap W=\phi$. Since f is semi-continuous, $f^{-1}(V)^\circ\neq\phi$ and $f^{-1}(W)^\circ\neq\phi$. Therefore, $f^{-1}(V)^\circ\cap f^{-1}(W)^\circ\subset f^{-1}(V)\cap f^{-1}(W)=\phi$, contradicting the fact that X is irreducible. Hence, Y must be irreducible.

COROLLARY 14. *The irresolute image of an irreducible space is irreducible.*

PROOF. Every irresolute map is semi-continuous.

COROLLARY 15. *Irreducibility is a semi-topological property, hence a topolo-*

gical property [3, Th. 1. 15].

THEOREM 16. *A topological space is irreducible if and only if every open filterbase accumulates to every point in X.*

PROOF. Let $F=\{0_a\}$ be an open filterbase. Let $x\in X$ be arbitrarily chosen. Then for every open V containing x, $O_a\cap V\neq\phi$ since X is irreducible. Therefore, $F\infty x$ for every $x\in X$.

Conversely, suppose every open filterbase accumulates to every $x\in X$. Let V, W be any two non-empty open sets. Choose $x\in V$, $y\in W$, and let $N(x)$ be the open neighborhood system about x. Then, since $N(x)$ is an open filterbase, $N(x)\infty y$. Since $V\in N(x)$, this implies $V\cap W\neq\phi$. Therefore, X is irreducible.

An S-connected space is connected in the usual sense since every open set is also semi-open. As noted earlier, an irreducible space X is connected. In Theorem 17, the irreducible spaces are shown to be precisely the S-connected spaces.

THEOREM 17. *In a topological space, the following are equivalent*:

1) *X is an irreducible space.*
2) *Any two semi-open sets have non-empty intersection.*
3) *X is S-connected.*
4) *There does not exist an irresolute function f from X onto $Y=\{a, b\}$ with the discrete topology.*

PROOF. $(1\to 2)$. If V and W are semi-open sets, they have non-empty interiors. Hence, $V^\circ\cap W^\circ\neq\phi$.

$(2\to 3)$. This is obvious.

$(3\to 4)$. If there exists an irresolute surjective function $f: X\to\{a,b\}$, then $X=f^{-1}(a)\cup f^{-1}(b)$ which implies X is not S-connected.

$(4\to 1)$. If X is not irreducible, then there exist disjoint non-empty open sets V and W in X. Since $V\cap W=\phi$, we know $V\cap\overline{W}=\phi$. Now $\overline{W}$ is semi-open and $(X-\overline{W})\supset V\neq\phi$. Therefore, if we define $f: X\to\{a, b\}$ by $f(x)=a$ if $x\in\overline{W}$ and $f(x)=b$ if $x\in X-\overline{W}$, we will have an irresolute function f.

THEOREM 18. *If X is irreducible, then X is S-closed.*

PROOF. Every open set is dense in X.

COROLLARY 19. *X is irreducible if and only if X is an S-continuum.*

COROLLARY 20. *The concept of S-continuum is a semi-topological property.*

and hence a topological property.

COROLLARY 21. *For a topological space X, the following are equivalent:*

1) *X is irreducible.*
2) *Every open filterbase accumulates to every point of x.*
3) *Every open set in X is dense.*
4) *X is S-connected.*
5) *S is an S-continuum.*

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