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AN APPLICATION OF THE FRACTIONAL DERIVATIVE I

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1. Introduction

There are many definitions of the fractional derivative. At first, J. Liouville [2] defined the fractional derivative of order α. Then, T. J. Osler defined the fractional derivative of order α in [5]. In 1974, B. Ross defined the fractional derivative of order α in [8]. Moreover, K. Nishimoto defined the fractional derivative and integral of order α in [4]. And in 1978, M. Saigo defined the integral operators in [9]. Furthermore in 1978, S. Owa gave the following definitions for the fractional derivative of order α in [6].

DEFINITION 1. The fractional derivative of order α is defined by

$$D_z^{\alpha} f(z)=\frac{1}{\Gamma(1-\alpha)} \frac{d}{dz}\int_0^z \frac{f(\zeta)d\zeta}{(z-\zeta)^{\alpha}},$$

where $0<\alpha<1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $ln(z-\zeta)$ to be real when $(z-\zeta)>0$. Moreover,

$$f(z)=\lim_{\alpha\to 0} D_z^{\alpha} f(z)$$

and

$$f'(z)=\lim_{\alpha\to 1} D_z^{\alpha} f(z).$$

DEFINITION 2. Under the hypotheses of Definition 1, the fractional derivative of order $(n+\alpha)$ is defined by

$$D_z^{n+\alpha} f(z)=\frac{d^n}{dz^n} D_z^{\alpha} f(z),$$

where $n\in N\cup\{0\}$.

Let S_0 denote the class of functions

$$f(z)=\sum_{n=1}^{\infty} a_n z^n$$

that are analytic and univalent in the unit disk U. And let $S_0^*(k)$ denote the class of functions

$$f(z)=\sum_{n=1}^{\infty} a_n z^n$$

that are starlike of order $k(0 \leqq k<1)$ with respect to the origin in the unit disk U. Furthermore, let $K_0(k)$ denote the class of functions

$$f(z)=\sum_{n=1}^{\infty} a_n z^n$$

that are convex of order $k(0 \leqq k<1)$ in the unit disk U. Then, we have

$$K_0(k) \subset S_0^*(k) \subset S_0.$$

2. A polynomial of degree at most n

M. A. Malik showed the following lemmas in [3].

LEMMA 1. *If $f(z)$ is a polynomial of degree at most n with $|f(z)| \leqq 1$ on $|z| \leqq 1$, then*

$$|f'(z)| \leqq n \qquad (|z| \leqq 1).$$

This result is best possible and equality holds for

$$f(z)=\alpha z^n$$

where $|\alpha|=1$.

LEMMA 2. *If $f(z)$ is a polynomial of degree at most n, with $|f(z)| \leqq 1$ on $|z| \leqq 1$, and $f(z)$ has no zero in the disk $|z|<K$, $K \geqq 1$, then for $|z| \leqq 1$,*

$$|f'(z)| \leqq \frac{n}{1+K}.$$

The result is best possible and equality holds for

$$f(z)=\left(\frac{z+K}{1+K}\right)^n.$$

In 1979, N. K. Govil, Q. I. Rahman and G. Schmeisser showed the following lemma in [1].

LEMMA 3. *If the function $f(z)$ is analytic and $|f(z)| \leqq 1$ in $|z| \leqq 1$, then*

$$|f(z)| \leqq \frac{(1-|a|)|z|^2+|bz|+|a|(1-|a|)}{|a|(1-|a|)|z|^2+|bz|+(1-|a|)} \quad (z \in U),$$

where $a=f(0)$ and $b=f'(0)$.

The example

$$f(z)=\frac{a+\frac{b}{1+a}z-z^2}{1-\frac{b}{1+a}z-az^2}$$

shows that the estimate is sharp.

THEOREM 1. *If*

$$f(z)=\sum_{k=1}^{n}a_kz^k$$

and $|f(z)|\leqq 1$ *in the unit disk* U, *then for* $0<\alpha<1$ *and* $z\in U$,

$$|D_z^{\alpha}f(z)|\leqq \frac{n|z|^{1-\alpha}}{\Gamma(2-\alpha)}$$

and

$$|D_z^{1+\alpha}f(z)|\leqq\frac{n(n+\alpha)}{\Gamma(2-\alpha)|z|^{\alpha}}.$$

PROOF. Let consider a polynomial

$$G(z)=\frac{\Gamma(2-\alpha)}{n}z^{\alpha}D_z^{\alpha}f(z).$$

Then, we have

$$|G(z)|\leqq|f(z)|\leqq 1$$

for $z\in U$. Using the Schwarz lemma, we have

$$|D_z^{\alpha}f(z)|\leqq\frac{n|z|^{1-\alpha}}{\Gamma(2-\alpha)}\qquad(z\in U).$$

Furthermore, by Lemma 1,

$$|G'(z)|\leqq n\qquad(z\in U).$$

Hence, we have

$$|D_z^{1+\alpha}f(z)|\leqq\frac{n(n+\alpha)}{\Gamma(2-\alpha)|z|^{\alpha}}\qquad(z\in U).$$

COROLLARY 1. *If*

$$f(z)=\sum_{k=1}^{n}a_kz^k$$

and $|f(z)|\leqq 1$ *in the unit disk* U, *then* $D_z^{\alpha}f(z)$ *is included in the disk with center at the origin and radius* $n/\Gamma(2-\alpha)$.

THEOREM 2. *If*

$$f(z)=\sum_{k=1}^{n}a_kz^k$$

and $|f(z)|\leqq 1$ *in the unit disk* U, *then for* $0<\alpha<1$ *and* $z\in U$,

$$|D_z^\alpha f(z)|\leqq\frac{|z|^{1-\alpha}(n|z|+|a_1|)}{\Gamma(2-\alpha)n(n+|a_1z|)}$$

and

$$|D_z^{1+\alpha}f(z)|\leqq\frac{1}{\Gamma(2-\alpha)|z|^\alpha}\left\{n^2+\frac{\alpha(n|z|+|a_1|)}{n(n+|a_1z|)}\right\}.$$

PROOF. Let

$$G(z)=\frac{\Gamma(2-\alpha)}{n}z^\alpha D_z^\alpha f(z).$$

Then, a polynomial $G(z)$ is analytic and $|G(z)|\leqq 1$ in the unit disk U. Therefore, we have

$$|G(z)|\leqq\frac{|z|(n|z|+|a_1|)}{(n+|a_1z|)},$$

that is,

$$|D_z^\alpha f(z)|\leqq\frac{|z|^{1-\alpha}(n|z|+|a_1|)}{\Gamma(2-\alpha)n(n+|a_1z|)}\qquad(z\in U)$$

with the aid of Lemma 3.

Moreover, using Lemma 1, we have

$$|G'(z)|\leqq n.$$

Hence, we have

$$|D_z^{1+\alpha}f(z)|\leqq\frac{1}{\Gamma(2-\alpha)|z|^\alpha}\left\{n^2+\frac{\alpha(n|z|+|a_1|)}{n(n+|a_1z|)}\right\}.$$

THEOREM 3. *Let*

$$f(z)=\sum_{k=1}^{n}a_kz^k$$

and $|f(z)|\leqq 1$ *in the unit disk* U. *If* $G'(z)$ *has no zero in the disk* $|z|<K(K\geqq 1)$, *then for* $0<\alpha<1$ *and* $z\in U$,

$$|D_z^{2+\alpha}f(z)|\leqq\frac{n}{\Gamma(2-\alpha)|z|^\alpha}\left\{\frac{n(n-1)}{1+K}+\frac{\alpha(2n+3\alpha-1)}{|z|}\right\},$$

where

$$G(z)=\frac{\Gamma(2-\alpha)}{n}z^\alpha D_z^\alpha f(z).$$

PROOF. Using the hypothesis of the theorem and Lemma 1,

$$|G'(z)| \leqq n$$

for $z \in U$. Therefore, $G'(z)/n$ is a polynomial of degree at most $(n-1)$, with $|G'(z)/n| \leqq 1$ in the unit disk U, and has no zero in the disk $|z| < K (K \geqq 1)$. Hence, with the aid of Lemma 2 and Theorem 1, we have the theorem.

3. On the special class of a polynomial

B. Pilat showed the following lemma in [7].

LEMMA 4. *A necessary and sufficient condition for*

$$f(z) = |a_1| z - \sum_{k=2}^{n} |a_k| z^k \in S_0$$

is

$$|a_1| - \sum_{k=2}^{n} k |a_k| = 0.$$

And, by some results were shown by H. Silverman in [10], we have immediately the following lemmas.

LEMMA 5. *A function*

$$f(z) = |a_1| z - \sum_{n=2}^{\infty} |a_n| z^n$$

is in the class $K_0(k)$ $(0 \leqq k < 1)$, if and only if

$$\sum_{n=2}^{\infty} n(n-k) |a_n| \leqq |a_1| (1-k).$$

LEMMA 6. *A function*

$$f(z) = |a_1| z - \sum_{n=2}^{\infty} |a_n| z^n$$

is in the class $S_0^*(k)$ $(0 \leqq k < 1)$, if and only if

$$\sum_{n=2}^{\infty} (n-k) |a_n| \leqq |a_1| (1-k).$$

LEMMA 7. *If the function*

$$f(z) = |a_1| z - \sum_{n=2}^{\infty} |a_n| z^n$$

is in the class $S_0^*(0)$, then for $z \in$ U,

$$\frac{|a_1||z|(2-|z|)}{2} \leqq |f(z)| \leqq \frac{|a_1||z|(2+|z|)}{2}.$$

Furthermore, N. K. Govil, Q. I. Rahman and G. Schmeisser showed the following lemma in [1].

LEMMA 8. *If*

$$f(z)=\sum_{k=1}^{3} a_k z^k$$

belongs to the class S_0, *then*

$$\max_{|z|=1} |f'(z)| \leqq \frac{3+2\sqrt{2}}{2+\sqrt{2}} \max_{|z|=1} |f(z)|.$$

THEOREM 4. *If*

$$f(z)=|a_1|z-\sum_{k=2}^{n} |a_k| z^k$$

belongs to the class S_0, *then for* $0<\alpha<1$ *and* $z \in U$,

$$\frac{|a_1||z|^{1-\alpha}(1-|z|)}{\Gamma(2-\alpha)} \leqq |D_z^\alpha f(z)| \leqq \frac{|a_1||z|^{1-\alpha}(1+|z|)}{\Gamma(2-\alpha)}.$$

PROOF. Since $f(z)$ is in the class S_0, using Lemma 4, we have

$$\begin{aligned}|\Gamma(2-\alpha)z^\alpha D_z^\alpha f(z)| &\leqq |a_1||z| + \sum_{k=2}^{n} \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} |a_k||z|^k \\ &\leqq |a_1||z| + |z|^2 \sum_{k=2}^{n} k|a_k| \\ &= |a_1||z|(1+|z|).\end{aligned}$$

In the same way, we have

$$|\Gamma(2-\alpha)z^\alpha D_z^\alpha f(z)| \geqq |a_1||z|(1-|z|).$$

These inequalities complete the proof of the theorem.

COROLLARY 2. *Under the hypotheses of Theorem 4,* $D_z^\alpha f(z)$ *is included in the disk with center at the origin and radius* $2|a_1|/\Gamma(2-\alpha)$.

THEOREM 5. *If*

$$f(z)=|a_1|z-|a_2|z^2-|a_3|z^3$$

belongs to the class $K_0(0)$, *then for* $0<\alpha<1$,

$$\max_{|z|=1} |D_z^{\alpha} f(z)| = \frac{|a_1|}{\Gamma(2-\alpha)}$$

and

$$\max_{|z|=1} |D_z^{1+\alpha} f(z)| \leqq \frac{|a_1| \{3+2\alpha+(2+\alpha)\sqrt{2}\}}{(2+\sqrt{2})\Gamma(2-\alpha)}.$$

PROOF. Let

$$G(z) = \Gamma(2-\alpha) z^{\alpha} D_z^{\alpha} f(z)$$
$$= |a_1| z - |A_2| z^2 - |A_3| z^3.$$

Then, by means of Lemma 5,

$$\sum_{k=2}^{3} k|A_k| \leqq \sum_{k=2}^{3} k^2 |a_k| \leqq |a_1|.$$

Hence, $G(z)$ is in the class $S_0^*(0) \subset S_0$ with Lemma 6. And, by Lemma 7, we have

$$\max_{|z|=1} |G(z)| = |a_1|,$$

that is,

$$\max_{|z|=1} |D_z^{\alpha} f(z)| = \frac{|a_1|}{\Gamma(2-\alpha)},$$

Furthermore, by means of Lemma 8,

$$\max_{|z|=1} |G'(z)| \leqq \frac{(3+2\sqrt{2})|a_1|}{2+\sqrt{2}},$$

that is,

$$\max_{|z|=1} |D_z^{1+\alpha} f(z)| \leqq \frac{|a_1| \{3+2\alpha+(2+\alpha)\sqrt{2}\}}{(2+\sqrt{2})\Gamma(2-\alpha)}.$$

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