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A CHARACTERIZATION OF COMPLEX SPACE FORMS AND ITS APPLICATION

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1. Introduction

Let M be a Riemannian manifold with Riemann-Christoffel curvature tensor R. Then, E. Cartan and J. A. Schouten obtained the following characterizations for the spaces of constant sectional curvature and of those which are (locally) conformal to Euclidean spaces.

THEOREM A [1]. *A Riemannian manifold M of dimension>2 is a real space form if and only if $R(X, Y; Z, X)=0$ for all orthonormal vectors X, Y and Z tangent to M at any of its points.*

THEOREM B [8]. *A Riemannian manifold M of dimension>3 is conformally flat if and only if $R(X, Y; Z, U)=0$ for all orthonormal vectors X, Y, Z and U tangent to M at any of its points.*

Let N be a Kaehlerian manifold with Riemann-Christoffel curvature tensor R. Then K. Yano and S. Sawaki obtained the following complex version of Theorem B which characterizes the spaces N with identically vanishing Bochner curvature tensor.

THEOREM C [10]. *A Kaehlerian manifold N of real dimension>6 is Bochner flat if and only if $R(X, Y; Z, U)=0$ for all orthonormal vectors X, Y, Z and U at any point p of N which span a totally real subspace of the tangent space T_pN.*

In Section 2 we prove the following similar complex version of Theorem A which characterizes the spaces of constant holomorphic sectional curvature.

THEOREM 1. *A Kaehlerian manifold N of real dimension>4 is a complex space form if and only if $R(X, Y; Z, X)=0$ for all orthonormal vectors X, Y and Z at any point p of N which span a totally real subspace of T_pN.*

The proof is based on the following theorem of B. Y. Chen and K. Ogiue.

THEOREM D [3]. *A Kaehlerian manifold of real dimension>4 is a complex space form if and only if it has constant anti-holomorphic sectional curvature.*

A Kaehlerian manifold N is said to satisfy the axiom of anti-holomorphic k-planes if for each $x\in N$ and each anti-holomorphic k-dimensional linear subspace π of T_xN there exists a k-dimensional totally geodesic submanifold M of N such that $x\in M$ and $T_xM=\pi$. The following is a theorem of K. Nomizu, B. Y. Chen and K. Ogiue.

THEOREM E [7] [3]. *A Kaehlerian manifold of dimension $2n$ satisfies the axiom of anti-holomorphic k-planes for some k, $2\leq k\leq n$, if and only if it is a complex space form.*

In Section 3, as an application of Theorem 1, we prove the following result which may be considered as some improvement of Theorem E.

THEOREM 2. *A Kaehlerian manifold N of real dimension $2n>4$ is a complex space form if and only if for every point p in N and every m-dimensional anti-invariant linear subspace T of T_pN, $2\leq m\leq n$, there exists a totally real m-dimensional submanifold M of N passing through p and having there T as tangent space such that M has commutative second fundamental tensors and parallel f-structure in the normal bundle.*

2. Proof of theorem 1

Let N be a Kaehlerian manifold of real dimension $2n>4$, with metric tensor g, complex structure J and Riemann-Christoffel curvature tensor R. Let X, Y and Z be orthonormal vectors which span an anti-invariant (or anti-holomorphic, or totally real) subspace S of the tangent space T_pN at an arbitrary point p. This means that the vectors JX, JY and JZ are perpendicular to S.

Then if N is a space of constant holomorphic sectional curvature c it follows that

$$R(X, Y; Z, X)=0, \tag{1}$$

since actually R is given by

$$\begin{aligned} R(A, B; C, D)=\frac{c}{4}\{&g(A,D)g(B,C)-g(A,C)g(B,D)+g(JA,D)g(JB,C)\\ &-g(JA,C)g(JB,D)+2g(A,JB)g(JC,D)\} \end{aligned} \tag{2}$$

for all vectors A, B, C and D tangent to N at each of its points.

Conversely we now assume that N satisfies (1) for all vectors X, Y and Z of the above type. Then also

$$R\left(X,\ \frac{Y+Z}{\sqrt{2}};\ \frac{Y-Z}{\sqrt{2}},\ X\right)=0, \tag{3}$$

which implies that

$$R(X,\ Y\ ;Y,\ X)=R(X,\ Z;\ Z,\ X). \tag{4}$$

Let U be any unit tangent vector at p which together with X determines a totally real plane, that is such that $g(X,\ U)=g(X,\ JU)=0$. We write U as

$$U=u_1U_1+u_2U_2 \tag{5}$$

whereby U_1 and U_2 are unit vectors belonging to the plane $Z\wedge JZ$ and its orthogonal complement in T_pN, respectively, and compute the sectional curvature for the plane section $X\wedge U$ making use of (1) and (4):

$$\begin{aligned} R(X,\ U\ ;U,\ X)&=R(X,\ u_1U_1+u_2U_2\ ;u_1U_1+u_2U_2,\ X) \\ &=u_1^2R(X,\ U_1\ ;U_1,\ X)+u_2^2R(X,\ U_2\ ;U_2,\ X) \\ &=u_1^2R(X,\ Y\ ;Y,\ X)+u_2^2R(X,\ Z\ ;Z,\ X) \\ &=(u_1^2+u_2^2)R(X,\ Y\ ;Y,\ X) \\ &=R(X,\ Y\ ;Y,\ X). \end{aligned} \tag{6}$$

This asserts that the sectional curvatures of N at p are equal for all totally real plane sections containing the vector X. Let V be any other unit tangent vector of N at p. Since $n>2$ we can always find a unit vector W in T_pN which is orthogonal to both X and Y and such that both planes $X\wedge W$ and $V\wedge W$ are anti-invariant. Then from (6) we have

$$R(V,\ W\ ;W,\ V)=R(X,\ W\ ;W,\ X), \tag{7}$$

and therefore may conclude that all totally real sectional curvatures of N at p are equal. By Theorem D this proves Theorem 1.

3. Proof of Theorem 2

Since the totally geodesic submanifolds M in the axiom of anti-holomorphic planes stated in Section 1 are automatically totally real submanifolds of the complex space form N [4] and since every totally geodesic totally real submanifold of a Kaehlerian manifold has commutative second fundamental tensors and parallel f-structure in the normal bundle [6], by Theorem E we need only to prove that the property

"*For every point p in a Kaehlerian manifold N of dimension $2n>4$ and every* *m-dimensional anti-invariant linear subspace T of T_pN for some m, $2\leq m\leq n$, there exists a totally real m-dimensional submanifold M of N with commutative second fundamental tensors and parallel f-structure in the normal bundle such that $p\in M$ and $T_pM=T$"*

implies that N has constant holomorphic sectional curvature.

To do so we first recall some well-known facts about m-dimensional totally real submanifolds M of a 2nd-dimensional Kaehlerian manifold N. The Kaehlerian metric, the corresponding Levi-Civita connection, the complex structure and the Riemann-Christoffel curvature tensor of N will be denoted by g, ∇, J and R. The induced Riemannian metric on M, the associated connection and curvature tensor will be denoted by $\bar{g}, \overline{\nabla}$ and $\overline{R}$. Because M is totally real the complex structure J maps every tangent vector of M into one which is normal to M in N, and so necessary $m\leq n$. The second fundamental form σ of M in N is defined by

$$\sigma(X, Y)=\nabla_X Y-\overline{\nabla}_X Y \tag{8}$$

where X and Y are arbitrary vector fields tangent to M. For a normal vector field ξ on M we write

$$\nabla_X\xi=-A_\xi X+D_X\xi \tag{9}$$

where $-A_\xi X$ and $D_X\xi$ are the tangential and the normal component of $\nabla_X\xi$. A_ξ is the second fundamental tensor of M with respect to ξ and D is the normal connection of M in N. We have

$$g(\sigma(X, Y), \xi)=\bar{g}(A_\xi X, Y). \tag{10}$$

The normal curvature tensor will be denoted by R^D, that is:

$$R^D(X, Y)=[D_X, D_Y]-D_{[X,Y]}. \tag{11}$$

Then the equations of Gauss, Codazzi and Ricci are given by [2]:

$$R(X, Y; Z, W)=\overline{R}(X, Y; Z, W)+g(\sigma(X, Z), \sigma(Y, W)) -g(\sigma(Y, Z), \sigma(X, W)), \tag{12}$$

$$(R(X, Y)Z)^{\perp}=(\nabla'_X\sigma)(Y, Z)-(\nabla'_Y\sigma)(X, Z), \tag{13}$$

$$R(X, Y; \xi, \varphi)=g(R^D(X, Y)\xi, \varphi)-\bar{g}([A_\xi, A_\varphi]X, Y), \tag{14}$$

where X,Y,Z and W are tangent vector fields, ξ and φ are normal vector fields, $(R(X, Y)Z)^{\perp}$ denotes the normal component of $R(X, Y)Z$ and ∇' is the connection of van der Waerden-Bortolloti:

(15) $$(\nabla'_X\sigma)(Y, Z)=D_X\sigma(Y, Z)-\sigma(\nabla_X Y, Z)-\sigma(Y, \nabla_X Z).$$

A normal section η is said to be ***cylindrical***, respectively ***geodesic***, if 0 is an eigenvalue of A_η with multiplicity $\geq m-1$, respectively if A_η vanishes identically. M is called a *totally cylindrical*, respectively a *totally geodesic submanifold* of N if there exist $2n-m$ mutually orthogonal normal sections on M which are cylindrical, respectively geodesic. M is said to be *geodesic with respect to a normal subbundle* $\mathscr{B}$ if every section in $\mathscr{B}$ is geodesic. The subbundle $\mathscr{B}^c$ of the normal bundle $T^\perp M$ which is orthogonal to a normal subbundle $\mathscr{B}$ and such that $\mathscr{B}\oplus\mathscr{B}^c=T^\perp M$ is called the *complementary subbundle* of $\mathscr{B}$. It is clear that the complementary subbundle $J(TM)^c$ of $J(TM)$ is holomorphic, that is invariant under J.

Let φ be any normal vector field on M in N. Following K. Yano and M. Kon [9], we put

(16) $$J\varphi=\mathrm{P}\varphi+f\varphi$$

where $P\varphi$ and $f\varphi$ are the tangential and normal component of $J\varphi$. Then P is a tangent bundle valued 1-form and f is an endomorphism of the normal bundle such that

(17) $$f^3+f=0.$$

Therefore if f doesn't vanish, that is if $m<n$, it defines an f-structure in $T^\perp M$. This structure is said to be *parallel* if for all tangent vector fields X and for all normal vector fields ξ we have

(18) $$(D_X f)\xi \overset{\text{def}}{=} D_X f\xi-fD_X\xi=0.$$

The f-structure in the normal bundle of a totally real submanifold M of a Kaehlerian manifold N is parallel if and only if M is geodesic with respect to the normal subbundle $J(TM)^c$ [6] [9].

If for all normal vector fields ξ and φ

(19) $$[A_\xi, A_\varphi]=0,$$

then we may choose a field of orthonormal frames $E_1, E_2, \cdots, E_m$ on M consisting of common eigenvectors of the second fundamental tensors A_ξ and such that

(20) $$A_{JE_i}E_j=\delta_{ij}h_jE_j$$

where δ_{ij} is a Kronecker delta [5]. This means that at most the j-th principal curvature h_j of M with respect to the normal section JE_j is non-zero, and thus that M is cylindrical with respect to the normal directions determined by an orthonormal frame of the normal subbundle $J(TM)$.

Consequently every totally real submanifold M with parallel f-structure in the normal bundle and commutative second fundamental tensors in a Kaehlerian manifold N is a totally cylindrical submanifold.

Now let N be a Kaehlerian manifold satisfying property (*) and let A, B and Q be any triple of orthonormal vectors which span an anti-holomorphic subspace of the tangent space at an arbitrary point p of N. Then by assumption there exists an m-dimensional totally real submanifold M of N passing through p for which $A, B \in T_pM$ and $Q \in J(T_pM)^c$ such that for every $\eta \in J(TM)^c$ we have

$$A_\eta = 0 \tag{21}$$

and with respect to the frame $E_1, E_2, \cdots, E_m$ choosen above we have (20). We'll prove the theorem by showing that

$$(R(A, B)A)^\perp = 0 \pmod{J(T_pM)}. \tag{22}$$

Indeed (22) implies in particular that

$$R(A, B ; A, Q) = 0 \tag{23}$$

which in view of Theorem 1 is equivalent to N being a complex space form. It is clear that if

$$(R(E_i, E_j)E_k)^\perp = 0 \pmod{J(TM)} \tag{24}$$

holds for all $i, j, k \in \{1, 2, \cdots, m\}$ then also (22) is true. Of course (24) is evident when $i=j$. Thus we must prove (24) in the following two cases: (I) i, j and k are mutually different; (II) $i=k\neq j$, or which amounts to the same: $j=k\neq i$. By (21) for all $X, Y \in TM$ we have [6]

$$\sigma(X, Y) = JA_{JY}X. \tag{25}$$

In case I (24) then follows at once from equation (13) of Codazzi and formula (20). In case II we consider the vector field $R(E_i, E_j)E_i$ whereby $i\neq j$. From (20) and (25) we find that

$$\sigma(E_i, E_j) = 0 \tag{26}$$

and

$$\sigma(E_i, E_i) = h_i JE_i. \tag{27}$$

Making use of (26) and (27) the equation of Codazzi yields

$$(R(E_i, E_j)E_i)^\perp = -h_i D_{E_j} JE_i \pmod{J(TM)}. \tag{28}$$

Finally, by (9) and the parallellism of the complex structure J, for any vector field $F \in J(TM)^c$ we have

$$g(D_{E_j} JE_i, F) = g(\nabla_{E_j} JE_i, F) \tag{29}$$
$$= g(J\nabla_{E_j} E_i, F)$$
$$= g(J\sigma(E_j, E_i), F)$$
$$= 0,$$

such that

$$(R(E_i, E_j)E_i)^{\perp} = 0 \ (\text{mod } J(TM)). \tag{30}$$

This ends the proof of Theorem 2.

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