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ON WEYL FRACTIONAL CALCULUS AND H-FUNCTION TRANSFORM

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1. Introduction

The Weyl fractional derivative of a function $g(z)$ is defined as follows:—

Suppose $g(z)\in A$, where A denotes a class of functions which are everywhere differentiable any number of times and if it and all of its derivatives are $0(x^{-r})$ for all r as $r\to\infty$ [4, p.82]. Then for $q<0$,

$$(1.1)\qquad {}_zD^q_\infty g(z)=\frac{(-1)^q}{\Gamma(-q)}\int_z^\infty (u-z)^{-q-1}g(u)du.$$

For $q\geq 0$,

$$(1.2)\qquad {}_zD^q_\infty g(z)=\frac{d^n}{dz^n}({}_zD^{q-n}_\infty g(z)),$$ n being a positive integer such that $n>q$.

The H-function transform of a function $f(t)$ is defined as [2, p.142]

$$(1.3)\qquad \bar{f}(p)=\int_0^\infty H^{M,N}_{P,Q}\left[(pt)^h\Big|{(a_i,\ \alpha_i)1,\ P \atop (b_i,\ \beta_i)1,\ Q}\right] f(t)dt,\ h>0,$$

where

$$(1.4)\qquad H^{M,N}_{P,Q}\left[z\Big|{(a_i,\ \alpha_i)\ 1,\ P \atop (b_i,\ \beta_i)\ 1,\ Q}\right]=\frac{1}{2\pi\omega}\int_L \theta(s)\ z^s ds,\ \omega=\sqrt{-1},$$

is the well-known H-function [3, p.594].

In (1.4),

$$(1.5)\qquad \theta(S)=(\prod_{i=1}^{M}\Gamma(b_i-\beta_i s)\prod_{i=1}^{N}\Gamma(1-a_i+\alpha_i s))x$$
$$\times(\prod_{i=M+1}^{Q}\Gamma(1-b_i+\beta_i s)\prod_{i=N+1}^{P}\Gamma(a_i-\alpha_i s))^{-1},$$

an empty product is interpreted as unity; the integers M, N, P, Q satisfy $1\leq M\leq Q$, $0\leq N\leq P$; α_i, β_i are all positive and the contour L is suitably chosen such that no poles of the integrand coincide.

If

$$(1.6)\qquad \lambda=\sum_1^M(\beta_i)-\sum_{M+1}^{Q}(\beta_i)+\sum_1^N(\alpha_i)-\sum_{N+1}^{P}(\alpha_i),$$

then the integral in (1.4) is absolutely convergent when

$$(1.7)\qquad \lambda>0,\ \ |\arg z|<\frac{1}{2}\lambda\pi.$$

The conditions of absolute convergence of the integral in (1.3) are given in [2].

By virtue of the relations [3, p. 600, (4.7)]

$$(1.8)\qquad H_{0,1}^{1,0}\left[t\,\middle|\,\begin{matrix}-\\(0,1)\end{matrix}\right]=e^{-t}$$

and

$(1,9)\qquad H_{1,1}^{1,1}\left[t\,\middle|\,\begin{matrix}(1-\alpha,\ 1)\\(0,1)\end{matrix}\right]=\Gamma(\alpha)(1+t)^{-\alpha}$, (1.3) reduces respectively to the classical Laplace transform given by

$$(1.10)\qquad g(p)=L[f(t);\ p]=\int_0^\infty e^{-pt}f(t)dt,$$

and

$(1.11)\qquad \bar{g}(p)=p^{-\alpha}\bar{f}\left(\frac{1}{p}\right)=\Gamma(\alpha)S_\alpha[f(t);\ p]$, which is the generalized Stieltjes transform [1, p. 233] given by

$$(1.12)\qquad S_\alpha[f(t);\ p]=\int_0^\infty(p+t)^{-\alpha}f(t)dt.$$

The aim of the paper is to establish a result for the H-function transform of $t^q f(t)$ (q real) in the Weyl fractional calculus as also to establish a generating relation for H-function with the help of generalized Taylor's formula [5].

THEOREM. *Suppose $\bar{f}(p)$, given by* (1.3), *be such that $\bar{f}(p)\in A$. Suppose the conditions* (1.6) *and* (1.7) *for the H-function hold. Then*

$$(2.1)\qquad (-1)^q_p D^q_\infty \bar{f}(p)=\phi[t^q f(t);\ p],\ \textit{for all } q \textit{ real},$$

where

$$(2.2)\qquad \phi[t^q f(t);p]=\int_0^\infty H_{P+1,Q+1}^{n+1,N}\left[(pt)^h\,\middle|\,\begin{matrix}\left(a_i-\frac{q\alpha_i}{h},\ \alpha_i\right)_{1,P},\ (-q,h)\\(0,h),\ \left(b_i-\frac{q\beta_i}{h},\ \beta_i\right)_{1,Q}\end{matrix}\right]\times t^q f(t)dt,$$

provided that the various integrals involved are absolutely convergent.

PROOF. Case I: For $q<0$, we have in view of (1.1) and (1.3),

$$(2.3)\qquad (-1)^q_p D^q_\infty \bar{f}(p)=\frac{1}{\Gamma(-q)}\int_p^\infty\left[(u-p)^{-q-1}\int_0^\infty H_{P,Q}^{M,N}\left[(ut)^h\,\middle|\,\begin{matrix}(a_i,\ \alpha_i)_{1,P}\\(b_i,\ \beta_i)_{1,Q}\end{matrix}\right]f(t)dt\right]du.$$

Inverting the order of integrations (which is justified under the conditions imposed with the theorem), we get

$$(2.4)\qquad (-1)^q {}_pD^q_\infty \bar{f}(p)=\frac{1}{\Gamma(-q)}\int_0^\infty f(t)\left[\int_p^\infty (u-p)^{-q-1} H^{M,N}_{P,Q}\left[(ut)^h \left| \begin{matrix}(a_i,\ \alpha_i)_{1,P}\\ (b_i,\ \beta_i)_{1,Q}\end{matrix}\right.\right] du\right] dt.$$

Since

$$(2.5)\qquad \int_p^\infty (u-p)^{-q-1} u^k H^{M,N}_{P,Q}\left[au^h \left| \begin{matrix}(a_i,\ \alpha_i)_{1,P}\\ (b_i,\ \beta_i)_{1,Q}\end{matrix}\right.\right] du$$

$$=p^{k-q}\Gamma(-q)\, H^{M+1,N}_{P+1,Q+1}\left[ap^h \left| \begin{matrix}(a_i,\ \alpha_i)_{1,P},\ (-k,\ h)\\ (q-k,\ h),\ (b_i,\ \beta_i)_{1,Q}\end{matrix}\right.\right]$$

where

$h>0$, $Re[k+h(a_i-1)/\alpha_i] < q < 0$ $(i=1, \cdots, N)$, $|\arg a| < \frac{1}{2}\lambda\pi$, $\lambda>0$ (λ being given by (1.6)), ((2.5) can be easily established), (2.4) reduces to

$$(2.6)\quad (-1)^q {}_pD^q_\infty \bar{f}(p)=\int_0^\infty p^{-q} H^{M+1,N}_{P+1,Q+1}\left[(pt)^h \left| \begin{matrix}(a_i,\ \alpha_i)_{1,P},\ (0,\ h)\\ (q,\ h),\ (b_i,\ \beta_i)_{1,Q}\end{matrix}\right.\right] f(t)dt.$$

By using [3, p. 596, eqns. (2.4) and (2, 2)], the result (2.1) is arrived at for $q<0$.

Case II: For $q\geqslant 0$, by utilising (1.2) and Case I, we can write

$$(2.7)\quad (-1)^q {}_pD^q_\infty \bar{f}(p)=(-1)^n \frac{d^n}{dp^n}$$

$$\left(\int_0^\infty H^{M+1,N}_{P+1,Q+1}\left[(pt)^h \left| \begin{matrix}\left(a_i-q'\frac{\alpha_i}{h},\ \alpha_i\right),\ (-q',\ h)\\ (0,h),\ \left(b_i-q'\frac{\beta_i}{h},\ \beta_i\right)_{1,Q}\end{matrix}\right.\right] t^{q'} f(t)dt\right),\ q'=q-n.$$

Differentiating under the sign of integral (valid under the conditions stated with the theorem), using [6, p. 131, (4.1)] and [3, p. 596, (2.4) and (2.2)], we find $(-1)^q {}_pD^q_\infty \bar{f}(p)=\phi[t^q f(t);p]$, for $q\geq 0$, where $\phi[-;p]$ is given by (2.2). This establishes the theorem for all real values of q.

3. Particular cases

1. On choosing parameters suitably as in (1.8), (2.1) reduces to a result due to authors [5 p. 189].

2. Similarly on choosing parameters as indicated in (1.9), replacing p by $1/p$, then the following known result [1, p. 213] emerges from (2.1), valid for all real q:

$$(3.1)\qquad (-1)^{q}\,{}_{p}D_{\infty}^{q}\,\bar{g}(p)=\Gamma(\alpha+q)S_{\alpha+q}[t^{q}f(t);\ p],$$

where $\bar{g}(p)$ is given by (1.11).

3. Because of the general nature of the H-functions, a number of other results can be derived from the main theorem on specialising parameters of the H-function in (2.1). These are omitted here for lack of space.

4. The generating relation

A generalization of the well-known Taylor's formula was recently given by the authors in the theory of Weyl fractional calculus in the following form [5, p. 189]:

Let $f(t)\in A$. Then,

$$(4.1)\qquad f(p+t)=\sum_{n=-\infty}^{\infty} c\times\frac{t^{cn+\eta}}{\Gamma(cn+\eta+1)}\,{}_{p}D_{\infty}^{cn+\eta}\ f(p),$$

where c is a real number such that $0<c\leq 1$, η is arbitrary and is valid for all t on the circle $|t/p|=1$.

In this section we use this result to establish a generating relation for H-function by taking $f(t)$ suitably.

We take

$$(4.2)\qquad f(t)=t^{a-1}H_{P,Q}^{M,N}\left[zt^{b}\,\middle|\,\begin{matrix}(a_i,\ \alpha_i)_{1,P}\\(b_i,\ \beta_i)_{1,Q}\end{matrix}\right],\ b>0$$

in (4.1) and use (2.5) etc. to get the generating relation:

$$(4.3)\qquad (p+t)^{a-1}H_{P,Q}^{M,N}\left[z(p+t)^{b}\,\middle|\,\begin{matrix}(a_i,\ \alpha_i)_{1,P}\\(b_i,\ \beta_i)_{1,Q}\end{matrix}\right]$$

$$=P^{a-1}\sum_{n=-\infty}^{\infty}\frac{c(-t/p)^{cn+\eta}}{\Gamma(cn+\eta+1)}H_{P+1,Q+1}^{M+1,N}\left[zp^{b}\,\middle|\,\begin{matrix}(a_i,\ \alpha_i)_{1,P},\ (1-a,\ b)\\(cn+\eta-a+1,\ b),\ (b_i,\ \beta_i)_{1,Q}\end{matrix}\right]$$

valid for $|t/p|=1$, $b>0$ and c being a real number such that $0<c\leq 1$ and η arbitrary.

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