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STANDARD HALFRINGS AND STANDARD IDEALS

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1. Introduction

The concept of a ring is derived from, and is a generalization of the algebraic properties of the set of integers. A semiring is a generalization of the algebraic properties of the set of nonnegative integers. Thus the concept of a semiring is much more general than that of a ring. Consequently, the class of semirings may be subdivided into a number of subclasses. One of the more interesting subclasses of semirings is the class of halfrings. Halfrings are those semirings capable of being embedded in a ring. Thus it is possible to move freely from the study of halfrings to the study of rings, and conversely. In this paper, we will look at the structure of halfrings, identify certain types of halfrings, and determine some of those properties of rings that are inherited by halfrings. We will also look at the relation between ideals in the halfring and ideals in the corresponding ring.

2. The classification of halfrings

A *semiring* is a nonempty set S together with two binary operations called addition $(+)$ and multiplication $(\cdot)$ such that $(S, +)$ is an abelian semigroup with a zero, $(S, \cdot)$ is a semigroup, and multiplication is distributive over addition from both the left and the right. We call a semiring S *commutative* if $ab=ba$ for all $a, b \in S$. A semiring with the cancellation property relative to addition will be called a *halfring*.

Let H be a halfring and $\overline{H}=H\times H$. In $\overline{H}$, define $(h, k)=(h', k')$ if and only if $h+k'=h'+k$. Since H has the cancellation property, it is an easy matter to show that equality in $\overline{H}$ is an equivalence relation and consequently, partitions $\overline{H}$. If we define addition and multiplication in $\overline{H}$ by

$$(h, k)+(h', k')=(h+h', k+k') \text{ and}$$
$$(h, k)(h', k')=(hh'+kk', hk'+kh')$$

then it can be shown that addition and multiplication are well defined and $\overline{H}$ is a ring with respect to these operations. Also, the mapping $\phi: H\rightarrow\overline{H}$ given by $h_{\phi}=(h, 0)$ is an injection and it follows that H is embedded in $\overline{H}$. If we identify the ordered pair (h, k) with $h-k$, then $\overline{H}=\{h-k \mid h, k \in H\}$ will be called the ring of differences of H. It is clear that $\overline{H}$ is the smallest ring containing H, i.e., the intersection of all rings containing H. We summarize this information in the following theorem.

THEOREM 2.1 (Embedding). *Any halfring H can be embedded in a ring. The smallest such ring containing H is the ring $\overline{H}$.*

We now want to identify certain types of halfring. To do this we will use the following definition.

DEFINITION 2.2. A semiring S is called a *strict semiring* if $a, b \in S$ and $a+b=0$ imply $a=b=0$. A *strict halfring* is defined similarly.

The set of nonnegative integers Z^+ is a strict halfring as well as the set $M_n(Z^+)$ of all $n \times n$ matrices over Z^+.

The strictness property enables us to characterize and investigate the structure of halfrings. Since every halfring contains a zero element 0 and $\{0\}$ is a ring with respect to the operations in H, it follows that every halfring contains a ring. When we say that a halfring H contains a ring R, we will mean $R \subset H$ and R is a ring with respect to the operations in H. The following theorem gives us a starting point.

THEOREM 2.3. *Let H be a halfring and R a ring such that $R \subset H$. Then H is strict if and only if $R=\{0\}$.*

PROOF. Suppose H is strict, $R \subset H$ and $a \in R$. Since R is a ring, $-a \in R$ and $a+(-a)=0$. From H being strict, it follows that $a=-a=0$. Consequently, $R=\{0\}$. On the other hand, suppose for any ring $R \subset H$ it follows that $R=\{0\}$ and suppose further that there are elements $a, b \in H$ with $a+b=0$. Since $a, b \in H \subset \overline{H}$, it follows that $b=-a \in H$. Consequently, $R'=\langle a, -a \rangle$, the set of all sums and products of the elements a and $-a$, is a ring contained in H. To see this, note that R' consists of all sums of elements of the form ma^t where $m \in Z$, Z the set of integers, and $t \geq 1$. Thus the elements of R' are of the form $\sum_1^k m_i a^i$ where $m_i \in Z$. It is clear that R' is closed under addition. Since $(ma^s)(na^t)=mna^{s+t}$, it follows that R' is closed under multiplication. Now $-a \in R'$ assures that the inverse of each element of R' is in R'. The distributive and associative properties are inherited from H. Consequently, R' is a ring contained in H. Therefore $R'=$

$\{0\}$ and it follows that $a=0$. Thus $a=b=0$ and H is strict.

Let H be a halfring, $\overline{H}$ its ring of differences, and $H^*=\{(0, b)\mid b\in H\}=\{-b\mid b\in H\}$. Either $H\cap H^*=\{0\}$ or $H\cap H^*\neq\{0\}$ and we have the following corollary.

COROLLARY 2.4. *A halfring H is strict if and only if $H\cap H^*=\{0\}$.*

PROOF. If $H\cap H^*\neq\{0\}$, then H contains a ring. For if $x\in H\cap H^*$, then x, $-x\in H$ and it follows from the proof of Theorem 2.3 that $J=\langle x, -x\rangle$ is a ring in H. Therefore Theorem 2.3 assures that H is strict if and only if $J=\{0\}$ and it follows that H is strict if and only if $H\cap H^*=\{0\}$.

With this information we can now classify a halfring according to whether or not it contains a nonzero ring. A halfring H will be called a *standard halfring* if H is strict. A halfring H will be called *semistandard* if H is not standard. It is possible to refine these two classifications even further. Note that if H is a halfring we always have $H\cup H^*\subset\overline{H}$. It may be that $H\cup H^*=\overline{H}$ or $H\cup H^*\neq\overline{H}$. We will call H a *type* I *halfring* if $\overline{H}=H\cup H^*$ and a *type* II *halfring* if $\overline{H}\neq H\cup H^*$. Consequently, our two major classifications of halfrings are standard and semistandard. Each of these classifications are refined to give subclasses of type I and type II. We now give examples of each of these classes.

EXAMPLES 2.5. (i) The set $H=Z^+$ of all nonnegative integers with the usual operations is a standard halfring of type I. It is clear that H is strict. For H^* is the set of all nonpositive integers, and it is clear that $H\cap H^*=\{0\}$ and $\overline{H}=H\cup H^*$.

Recall that an *ordered ring* is a ring R together with a subset P of R such that

(1) $0\notin P$,

(2) if $a\in R$ then either $a\in P$, $a=0$, or $-a\in P$,

and (3) if $a, b\in P$ then $a+b$ and $ab\in P$.

If we put $0\in P$ and let $P^*=\{-a\mid a\in P\}$, then P is a standard halfring of type I. For clearly, $R=\overline{P}=P\cup P^*$ and $P\cap P^*=\{0\}$. Consequently, any ordered ring contains a standard halfring of type I.

(ii) The set $K=M_2(Z^+)$ of all 2×2 matrices over Z^+ is a standard halfring of type II. It is rather obvious that K is strict. Now K^* is set of 2×2 matrices with nonpositive entries and $\overline{K}=M_2(Z)$. Also we have $K\cap K^*=\{0\}$ where 0 is the zero matrix. Since there are matrices in $\overline{K}$ with both negative and positive entries, it follows that $\overline{K}\neq K\cup K^*$.

(iii) The set $H=Z^+\times Z$ with the usual componentwise operations of addition and multiplication is a semistandard halfring of type I. Clearly $\overline{H}=Z\times Z$, $H^*=N\times Z$, where N is the set of all nonpositive integers, and $H\cap H^*=\{(0, n) \mid n\in Z\}\cong Z$ is a nonzero ring contained in H. Also, it is easy to see that $\overline{H}=H\cup H^*$.

(iv) The set $H=Z^+\times Z^+\times Z$ with the usual componentwise operations of addition and multiplication is a semistandard halfring of type II. Now $\overline{H}=Z\times Z\times Z$, $H^*=N\times N\times Z$, where N is the set of all nonpositive integers, and $H\cap H^*=\{(0,0,n)\mid n\in Z\}\cong Z$ is a nonzero ring contained in H. It can be easily shown that $\overline{H}\neq H\cup H^*$.

A few remarks about standard halfrings. If H is a standard halfring of type I, then $\overline{H}$ is an ordered ring with the set P being $H-\{0\}$. If H and K are standard halfrings of type I, then $H\times K$ is a standard halfring of type II. A standard halfring H of type I is nice to work with since $H\cap H^*=\{0\}$ and $\overline{H}=H\cup H^*$. For the remainder of this paper we will assume that standard halfring will mean standard halfring of type I.

3. Ideals in a standard halfring

If H is a commutative halfring with an identity, then it follows that $\overline{H}$ is a commutative ring with an identity. In what follows we will assume that all halfrings are commutative with an identity.

DEFINITION 3.1. A nonempty subset I of a halfring H is called an *ideal* of H if for $a, b\in I$ and $h\in H$

(i) $a+b$ and $ab\in I$,

and (ii) ha and $ah\in I$.

Now there is a natural relation between ideals in a halfring H and ideals in the ring $\overline{H}$. If J is an ideal in $\overline{H}$ then it is straight forward to show that $J\cap H$ is an ideal in H. On the other hand, if A is an ideal in H, we want to relate A to some ideal in $\overline{H}$. Since A is a subset of $\overline{H}$ the ideal $A'=\bigcap\{I\mid I$ is an ideal in $\overline{H}$ and $A\subset I\}$ is an ideal containing A. Now let $\overline{A}=\{a-b\mid a, b\in A\}$.

THEOREM 3.2. *If A is an ideal in H, then $\overline{A}$ is an ideal in $\overline{H}$.*

PROOF. Suppose $x, y\in\overline{A}$ and $\alpha\in\overline{H}$. Now $x=a-b$, $y=c-d$ for some a, b, c, d

$\in A$, and it follows that

$$x-y=(a-b)-(c-d)=(a+d)-(b+c)\in\overline{A} \text{ and}$$
$$xy=(a-b)(c-d)=(ac+bd)-(bc+ad)\in\overline{A}.$$

Also $\alpha=p-q$ for some p, $q\in H$ and it follows that

$$\alpha x=(p-q)(a-b)=(pa+qb)-(qa+pb)\in\overline{A}.$$

This is clear since A is an ideal in H. Thus $\overline{A}$ is an ideal in $\overline{H}$.

Since $A\subset\overline{A}$ it follows that $A'\subset\overline{A}$. But any ideal containing A must contain $\overline{A}$. Thus $\overline{A}\subset A'$ and it follows that $\overline{A}=A'$. Therefore we can associate the ideal $A\subset H$ with the ideal $\overline{A}\subset\overline{H}$. Now it is not generally true that if A is an ideal in H, then $A=\overline{A}\cap H$. To see this, let $H=Z^+$ and $A=\{0, 6, 8, 10, \cdots\}$. Now $\overline{A}=\langle 2\rangle$, the ideal in Z generated by 2, and $\overline{A}\cap H=\overline{A}\cap Z^+=\{0, 2, 4, 6, \cdots\}\neq A$. Thus it may happen that A and B are ideals in H with $\overline{A}=\overline{B}$ but $A\neq B$.

DEFINITION 3.3. An ideal A in a halfring H is called a *standard ideal* if $A=\overline{A}\cap H$.

We note that if H is a standard halfring and J is an ideal in $\overline{H}$, then $J\cap H$ is a standard ideal in H. Recall from the literature that an ideal I in a semiring S is called a *k-ideal* (*subtractive ideal*) if a, $a+b\in I$ and $b\in S$ imply $b\in I$. In a halfring, the notions of standard ideal and k-ideal are equivalent.

THEOREM 3.4. *Let H be a standard halfring and A an ideal in H. Then A is a standard ideal if and only if A is a k-ideal.*

PROOF. If A is a standard ideal and a, $a+b\in A$ with $b\in H$, then $b=(a+b)-a\in\overline{A}$. Consequently, $b\in\overline{A}\cap H=A$ and it follows that A is a k-ideal. Conversely, suppose that A is a k-ideal in H. Clearly, $A\subset\overline{A}\cap H$. Assume $x\in\overline{A}\cap H$. Then $x\in\overline{A}$ and $x\in H$. Now $x\in\overline{A}$ gives $x=a-b$ where $a, b\in A$. Thus $a=b+x$ and A being a k-ideal assures that $x\in A$. Consequently, $\overline{A}\cap H\subset A$ and it follows that $A=\overline{A}\cap H$ and A is a standard ideal.

If A is a standard ideal in a standard halfring H, then from $A=\overline{A}\cap H$ it follows that $\overline{A}-A\subset\overline{H}-H$. We will now look at some of the properties of A carried over to $\overline{A}$ and some of the properties of A inherited from $\overline{A}$.

THEOREM 3.5. *Let H be a standard halfring and P a standard ideal in H. Then P is prime if and only if $\overline{P}$ is prime in $\overline{H}$.*

PROOF. Suppose P is prime in H and $ab\in\overline{P}$. Since H is standard, either $ab\in P$ or $ab\in\overline{P}-P$. If $ab\in P$ the theorem follows easily. If $ab\in\overline{P}-P\subset\overline{H}-H$, then $-ab \in H$. Thus $-ab \in \overline{P} \cap H = P$. Hence either $-a \in P$ or $b \in P$ and it follows that $-(-a) = a \in \overline{P}$ or $b \in P \subset \overline{P}$. Consequently, $\overline{P}$ is a prime ideal in $\overline{H}$. Conversely, suppose that $\overline{P}$ is a prime ideal and $c, d \in H$ such that $cd \in P = \overline{P} \cap H$. Then $c \in \overline{P}$ or $d \in \overline{P}$ and it follows that $c \in \overline{P} \cap H = P$ or $d \in \overline{P} \cap H = P$. Consequently, P is a prime ideal in H.

THEOREM 3.6. *Let H be a standard halfring and M a standard ideal in H. Then M is maximal with respect to standard ideals if and only if $\overline{M}$ is maximal in $\overline{H}$.*

PROOF. Suppose $\overline{M}$ is maximal in $\overline{H}$ and N is a standard ideal such that $M \subset N \subset H$. Then $\overline{M} \subset \overline{N} \subset \overline{H}$ and it follows that $\overline{M} = \overline{N}$ or $\overline{N} = \overline{H}$. If $\overline{M} = \overline{N}$, then $M = \overline{M} \cap H = \overline{N} \cap H = N$. If $\overline{N} = \overline{H}$, then $N = \overline{N} \cap H = \overline{H} \cap H = H$. Consequently, M is maximal with respect to standard ideals. Now suppose that Q is maximal with respect to standard ideals in H and J is an ideal in $\overline{H}$ such that $\overline{Q} \subset J \subset \overline{H}$. Then $J' = J \cap H$ is a standard ideal in H and $Q \subset J' \subset H$. Since Q is maximal with respect to standard ideals, we have $J' = Q$ or $J' = H$. Hence $\overline{Q} = \overline{J'} = J$ or $\overline{J'} = \overline{H} = J$ and it follows that $\overline{Q}$ is maximal in $\overline{H}$.

Standard is necessary in the above theorem. For it can happen that M is maximal with respect to standard ideals and there exists an ideal I such that I is not a standard ideal but $M \subset I \subset H$ is a proper inclusion. When this happens it follows that $\overline{I} = \overline{H}$.

It is straight forward to show that any principal ideal in H is a standard ideal. Also, if P is a principal ideal in $\overline{H}$ then $P \cap H$ is a principal ideal in H. Thus, it can be shown that if P is a standard ideal in H, then P is principal if and only if $\overline{P}$ is principal. We now know that the prime, maximal, and principal properties of ideals in a standard halfring are carried over to related ideals in the ring of differences. Conversely, these same properties of ideals in the ring of differences are inherited by standard ideals in the halfring. Further, if $\overline{H}$ is principal ideal ring, it is not true in general that H is a principal ideal halfring. Also if H is a local ring, it is not true in general that $\overline{H}$ is a local ring.

4. Homomorphisms of halfrings

In this section we will consider homomorphisms of standard halfrings.

THEOREM 4.1. *Let $f: H \to K$ be a homomorphism of the halfrings H and K. Then f can be extended to a homomorphism $\overline{f}: \overline{H} \to \overline{K}$ of the rings $\overline{H}$ and $\overline{K}$.*

PROOF. Let $f: H \to K$ be a homomorphism. Now any $x \in \overline{H}$ has the form $x = a - b$ where $a, b \in H$. Define $\bar{f}: \overline{H} \to \overline{K}$ by $\bar{f}(x) = f(a) - f(b)$. Since $f(a), f(b) \in K$, it is clear that $\bar{f}(x) \in \overline{K}$. Now suppose $x = a - b$ and $x = c - d$. Then $a + d = b + c$ and f being a homomorphism assures that $f(a) + f(d) = f(b) + f(c)$ and consequently $f(a) - f(b) = f(c) - f(d)$. Thus $\bar{f}$ is well defined. Now if $y = a' - b'$, then

$$\begin{aligned}
\bar{f}(x+y) &= \bar{f}[(a-b)+(a'-b')] \\
&= \bar{f}[(a+a')-(b+b')] \\
&= f(a+a')-f(b+b') \\
&= f(a)+f(a')-\{f(b)+f(b')\} \\
&= \{f(a)-f(b)\}+\{f(a')-f(b')\} \\
&= \bar{f}(x)+\bar{f}(y)
\end{aligned}$$

and

$$\begin{aligned}
\bar{f}(xy) &= \bar{f}[(a-b)(a'-b')] \\
&= \bar{f}[(aa'+bb')-(ab'+a'b)] \\
&= f(aa'+bb')-f(ab'+a'b) \\
&= [f(a)f(a')+f(b)f(b')]-[f(a)f(b')+f(a')f(b)] \\
&= [f(a)-f(b)][f(a')-f(b')] \\
&= \bar{f}(x)\bar{f}(y).
\end{aligned}$$

Consequently, $\bar{f}$ is a ring homomorphism. It is clear that $\bar{f}$ restricted to H is f. Thus $\bar{f}$ is an extension of f.

From Theorem 4.1 we can derive a couple of other results. If $f: H \to K$ is a homomorphism, then it is straightforward to show that $\bar{f}(H) = \overline{f(H)}$. Also if f is surjective and J is a standard ideal in K, then $f^{-1}(J)$ is a standard ideal in H. It is also valid that the homomorphic image of a standard halfring is standard.

In all examples of standard halfrings of type II, I have found that these were the sum of a halfring and a ring. I have not been able to prove this. I would like to close this paper with the following question. Is it true that H is a standard halfring of type II if and only if $H = H' + R$ where H' is a standard halfring of type I and R is a ring?

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